Statistics 530: Probability Theory
Homework No. 8

READING

Press on with your reading of Durrett through page 158.

PROBLEM 1. QUICK SHOTS. For $\lambda > 0$, justify the following assertions:

- $\phi_1(t) = \exp(\lambda(e^{it} - 1))$ is a characteristic function.
- $\phi_2(t) = \exp(\lambda(1 - e^{it}))$ is NOT a characteristic function.
- $\phi_3(t) = \exp(\lambda(\cos t - 1))$ is a characteristic function.

PROBLEM 2. Suppose that $x_n = (x_{n,1}, x_{n,2}, \ldots, x_{n,d})$ is a sequence of vectors in $\mathbb{R}^d$. Suppose that $x_{n,j} \geq 0$ for all $n \geq 1$ and $1 \leq j \leq d$, and suppose that there is an A such that for $n \to \infty$,

$$x_{n,1} + x_{n,2} + \cdots + x_{n,d} \to dA$$
$$x_{n,1}^2 + x_{n,2}^2 + \cdots + x_{n,d}^2 \to dA^2$$

Show that $(x_{n,1}, x_{n,2}, \ldots, x_{n,d}) \to (A, A, ..., A)$ as $n \to \infty$.

Hint: This is a cool real variable fact, but why would I pester you with it at this point in your life? It is because this elementary fact illustrates the principle used to prove Levy's continuity theorem. For the *determining condition* consider the possibility that we had equality in these two relations (not just limits). Show how the case of equality in Cauchy-Schwarz (or the case of equality in Jensen's inequality) would then imply that $(x_{n,1}, x_{n,2}, \ldots, x_{n,d})$ is a constant vector. Now use compactness and subsequence arguments to get the conjectured limit theorem.

PROBLEM 3. Suppose that $f_n(x)$ is a sequence of densities such that $f_n(x)$ converges to $f(x)$ for all $x \in \mathbb{R}$. Show that if f is *also* a density, then

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |f_n(x) - f(x)|\, dx \to 0 \quad \text{as } n \to \infty.$$

HINT: Even more than usual, you have to use all of the hypotheses! Bigger hint — What can you say about the sequence of functions

$$g_n(x) = \{f(x) - f_n(x)\}1(x : f(x) - f_n(x) \geq 0\}$$

NOTE: This is a problem we could have discussed in the first couple of weeks. It is left until now because earlier we ourselves did not give ourselves licence to think about densities. It is a good discipline to avoid them whenever it is sensible to do so.

PROBLEM 4. Complete the outline below to show that if for each $k = 0, 1, 2, ...$ the sequence of random variables Z_n satisfies

$$E(Z_n^k) \to \frac{1}{1+k} \quad \text{as } n \to \infty$$

then Z_n converges in distribution to the uniform distribution on $[0, 1]$ as $n \to \infty$. Here is the outline:

- Show that the sequence of distributions of the Z_n is tight.

- Show that if $E(W^k) = \frac{1}{1+k}$ for all $k = 0, 1, ...$ then W is uniformly distributed on $[0, 1]$. Please confine your tools to results that we have proved in class. No magic incantations, SVP.
- Follow the pattern we used to prove Levy's continuity theorem to complete the solution of the problem. You will probably want to use a subsequence argument and perhaps an argument by contradiction.

NOTE: By understanding this exercise and related exercises, you will get to understand the pattern behind Levy's continuity theorem. This is an admission ticket to a reasonably small club, much smaller than the Masons, for example.

PROBLEM 5. Suppose that X is a random variable with $P(X > 0) = 1$ and with a finite mean $\mu > 0$. Complete the following plan:

- Explain why $f(x) = \mu^{-1}P(X > x)$ is a probability density on $[0, \infty)$.
- Suppose that Y is a random variable with density f. Calculate the characteristic function $\psi(t)$ of Y in terms of the characteristic function of $\phi(t)$ of X. Note: The answer is a simple algebraic function of μ, $\phi(t)$, and constants. You will need to integrate by parts.
- Check your formula by confirming directly from the formula that $\psi(0) = 1$.
- Suppose X has the exponential distribution with mean one. What is the distribution of Y?

NOTE: This is a very general recipe and — more often than one might suspect — the examination of Y will tell us something interesting about the distribution of X. Here for example, the uniform continuity of $\psi(t)$ gives us some additional "continuity information" about $\phi(t)$ in a form that one could never conjectured without this auxiliary calculation.

It is very handy to have "standard recipes" that tell us how to morph one problem into another. You should keep a list of these for future uses. Sometimes such morphings just send us around in circles, but at luckier times we end up discovering something amusing — or even genuinely new.