Statistics 930: Probability Theory
Homework No. 11

General Comments: Some of these are intuitive, foundational problems where it is easy to just make a bunch of assertions that are reasonable and that end with the desired conclusion. To really ring the bell, one has to be brutally attentive to the logic, and, if you say the word "obvious," then you have almost surely made a mistake. In this context, the best proofs tend to have the fewest words and the tightest logic.

Problem 1. Suppose $\{\mathcal{F}_n\}$ is an increasing sequence of σ-fields, and set

$$\mathcal{P} = \cup \mathcal{F}_n \quad \text{and} \quad \mathcal{F} = \sigma\{\cup \mathcal{F}_n\} = \sigma\{\mathcal{P}\}.$$

Given a probability measure P on $\mathcal{F}$, we then consider the set

$$\mathcal{G} = \{A \in \mathcal{F} : \forall \epsilon > 0, \, \exists A_\epsilon \in \mathcal{P}, \text{ such that } P(A \Delta A_\epsilon) \leq \epsilon\}.$$

Show that $\mathcal{G}$ is a λ-system and use this to deduce that $\mathcal{F} \subset \mathcal{G}$. In longhand, this says that any element of $\mathcal{F}$ can be approximated as well as we like by an element of $\mathcal{P}$.

Problem 2. Suppose that $\{X_1, X_2, \ldots\}$ is an infinite sequence of (possibly dependent) random variables and let $\mathcal{F}$ be the smallest sigma-field for which these are all measurable. Suppose X is a bounded, $\mathcal{F}$-measurable random variable. Show that there for each $\epsilon > 0$, there is an n and a function $f : \mathbb{R}^n \to \mathbb{R}$ such that

$$\mathbf{E}[|X - f(X_1, X_2, \ldots, X_n)|] \leq \epsilon.$$

Remark. You probably want to make use of Problem 1.

Problem 3. Consider sequences of random variables such that X_n converges in probability to 1 as $n \to \infty$ and $\mathbf{E}[Y_n] = \alpha$ for all $n = 1, \ldots.$

- Give an example such that the product is $X_n Y_n$ is integrable for each n, but the sequence $\mathbf{E}[X_n Y_n]$ does not converge to α as $n \to \infty$.
- Now assume that the collection $\{X_n, Y_n, X_n Y_n : n = 1, 2, \ldots\}$ is uniformly integrable, and show that $\mathbf{E}[X_n Y_n]$ converges to α as $n \to \infty$. Examine your proof to see if you can weaken the uniform integrability condition.

Problem 4.

- Let C be the collection of functions $f : \Omega \to [0, 1]$. This a convex set, and its easy to see that the extreme points are those functions that take values in the two-point set $\{0, 1\}$, but this not too useful in probability theory. State and prove a (slightly) better version for a probability space with $\Omega = [0, 1]$.
- Let C be the set of probability measures on the Borel subsets of $[0, 1]$. For each x let μ_x be the probability measure such that $\mu_x(\{x\}) = 1$. Show that for each x the measure μ_x is an extreme point of C and show that every extreme point of C can be written as μ_x for some x. If you can't prove the general result, prove the best result you can.
- Let C be the set of twice differentiable, convex functions $f : [0, 1] \to [0, 1]$ such that $f(0) = 1$ and $f(1) = 0$. Show that C is convex, and show that $f(x) = 1 - x$ is the only extreme point of C. This gives us a pretty big set with just one extreme point, but this may not be too surprising since we saw in class that the unit ball in $L^1[0, 1]$ has no extreme points.