**Statistics 530: Probability Theory Midterm Fall 2010**

INSTRUCTIONS.

- Read all of the instructions.
- When the time is up, stop working and hand in your blue book (or blue books) — without exception or delay.
- If a problem X is not done, write "Problem X was not done" in the blue book.
- Be careful to avoid any embarrassing situations. Keep your paper out of the eyesight of others and keep your eyes away from the work area of others.
- Write clearly. Check your work.
- If you have more than one blue book number them "1 of 3", "2 of 3", etc.
- Be sure that your blue books have your printed name and your signature.
- Notation: $1_A$ or $1(A)$ are both used to denote the indicator function of the event $A$, i.e. a random variable that equals 1 for $\omega \in A$ and equals zero otherwise.
- Generic Question: "Can I use ...". Generic Answer: "You can use ANY result that has been proved in class and ANY result from real analysis."
- Generic Advice: **Do not get stuck! If you don't have an idea for a problem, or if your idea is not working, go to another problem.**

PROBLEM 1.

- State the first Borel-Cantelli Lemma
- State the second Borel-Cantelli Lemma
- State Hölder's inequality
- State Markov's inequality
- State Chebyshev's inequality

PROBLEM 2. Suppose that $\{X_n\}$ is a sequence of random variables on the probability space $(\Omega, \mathcal{F}, P)$. such that

$$P(|X_n| \geq t) \leq a_n/t \quad \text{for all } t > 0,$$

where $\{a_n\}$ is a sequence of real numbers such that

$$\sum_{n=1}^{\infty} a_n^2 < \infty.$$

Show that $a_n X_n$ converges to zero with probability one.

PROBLEM 3. Suppose that $\{X_k\}$ is a sequence of random variables on the probability space $(\Omega, \mathcal{F}, P)$ such that all of the random variables of the sequence have the same distribution, but the random variables are not necessarily independent. Suppose that we have

$$E(X_k) = 0 \quad \text{and} \quad E(X_k^4) = 1 \quad \text{for all } k \geq 1.$$

Let $Y_k = X_k 1(|X_k| \leq k^\alpha)$ and find the smallest $\alpha_0$ that you can such that for all $\alpha > \alpha_0$ the event

$$A = \{\omega : X_k(\omega) = Y_k(\omega) \text{ for all } k \text{ except for finitely many exceptions }\}$$

has probability one.

Problem 4. Suppose that $\{X_n\}$ and $\{Y_n\}$ are two sequence of random variables on the probability space $(\Omega, \mathcal{F}, P)$. The sequence $\{X_n\}$ is i.i.d with $EX_n = 0$ and $EX_n^2 = 1$ for all $n \geq 1$. The sequence $\{Y_n\}$ is i.i.d with $EY_n^4 = 1$ for all $n \geq 1$. The sequences $\{X_n\}$ and $\{Y_n\}$ are not necessarily independent of each other. Show that

$$R_n = \frac{X_1 + X_2 + \cdots + X_n}{Y_1^2 + Y_2^2 + \cdots + Y_n^2}$$

converges to zero with probability one.

Problem 5. Consider a non-negative random variable $X$ on the probability space $(\Omega, \mathcal{F}, P)$ such that

$$E[X1(X \geq t)] \leq 2/t \quad \text{for all } t \geq 0.$$

Show that for each event $A \in \mathcal{F}$ we have the bound

$$E[X1_A] \leq 2\sqrt{2P(A)}.$$

Problem 6. Suppose that $\{X_n\}$ is a sequence of i.i.d. random variables on the probability space $(\Omega, \mathcal{F}, P)$. Assume that

$$E[|X_1|] < \infty \quad \text{and} \quad E[X_1] = 0.$$

Show that the random variable

$$Z_n = \frac{1}{n} \sum_{k=1}^{n} X_k X_{k+1}$$

converges to zero with probability one.

Final Advice. Check your work. Did you use the hypotheses? At any place did you actually assume more that was given? For each of the problems (except the first) you can give a 100% correct proof on about one-half of one blue book page — certainly one blue book page should suffice. If there is some technical issue that is bothering you, it probably should not — or perhaps you don't have the right plan. You can make a note about what is worrying you and then return to that point later if you have time.