NOTES on π-λ Theorem. JMS 11/14/2016

π-systems and λ-systems and σ-fields (Defn and Basic Properties)

$\mathcal{H}$ is a π-system $\Leftrightarrow$ A, B in $\mathcal{H}$ $\Rightarrow A \cap B \in \mathcal{H}$

i.e. closed under intersection

Stronger "all together"

λ-system $\mathcal{H}$	σ-field $\mathcal{F}$
1. $\Omega \in \mathcal{H}$	1'. $\Omega \in \mathcal{F}$
2. $A \subset B$, $A \in \mathcal{H}$, $B \in \mathcal{H}$ $\Rightarrow B \setminus A \in \mathcal{H}$	2'. $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$
3. $A_n \subset A_{n+1} \Rightarrow \bigcup^{\infty} A_n \in \mathcal{H}$	3'. $A_n \in \mathcal{F} \Rightarrow \bigcup A_n \in \mathcal{F}$

$\boxed{\sigma\text{-field criterion}}$

~~[illegible]~~

If $\mathcal{H}$ is a π-system and a λ-system then $\mathcal{H}$ is a σ-field.

Proof. #1' and #2' field properties are free!

Now suppose $A_1, A_2 \ldots$ are given.

Start with A, B elements of $\mathcal{H}$ we have $(A \cup B)^c = A^c \cap B^c \in \mathcal{H}$ since $A^c, B^c \in \mathcal{H}$, $\mathcal{H}$ π-sset

Hence $A_1 \cup A_2$, $A_1 \cup A_2 \cup A_3 \ldots$ are in $\mathcal{H}$ $A_n' = A_1 \cup \cdots \cup A_n$, $A_n' \subset A_{n+1}'$

$\bigcup A_n' = \bigcup A_n$ and $\bigcup A_n'$ in $\mathcal{H}$ so $\bigcup A_n$ in $\mathcal{H}$ (3')

minimal systems.

Defn $\mathcal{C}$ collection of sets:

$\sigma(\mathcal{C})$ smallest σ-field containing $\mathcal{C}$

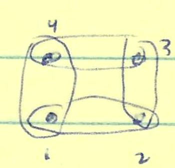
$\lambda(\mathcal{C})$ smallest λ-system containing $\mathcal{C}$

Proof the latter exists: Consider the intersection of all λ-systems containing $\mathcal{C}$.

#1 and #2 — and #3 are trivial.

Examples and Illustrations.

Ex 1. $\Omega = \{1,2,3,4\}$



"cycle λ-system"

$\mathcal{H} = \{ \Omega, \emptyset, \{1,2\}, \{2,3\}, \{3,4\}, \{4,1\} \}$

$\mathcal{H}$ is a λ-system but $\mathcal{H}$ is not a π-system and $\mathcal{H}$ is not a σ-field.

Ex 2. Suppose μ_1 and μ_2 are two prob measures on $(\Omega, \mathcal{F})$

Let $\mathcal{H} = \{ A : \mu_1(A) = \mu_2(A) \}$

"equality λ-system"

$\mathcal{H}$ is a λ system: (1) is trivial for (2) $A \subset B$

$$\begin{array}{l} \mu_1(B) = \mu_2(B) \\ \underline{\mu_1(A) = \mu_2(A)} \end{array}$$

$$\mu_1(A) - \mu_2(A) = \mu_1(B) - \mu_2(B) \Rightarrow \mu_1(B \setminus A) = \mu_2(B \setminus A)$$

For (3) we have the MCT

Ex 3. $\mathcal{P} = \{ (-\infty, a] : a \in \mathbb{R} \}$ $\quad \mathcal{P}$ is a π-system.

"interval π-system"

also $\lambda(\mathcal{P})$ is the Borel σ-field.

— not, obviously not a λ-system, or a σ-field

Note: Ex 2 and Ex 3 are highly representative.

Dynkin's Theorem and Application.

$\boxed{\text{Dynkin: If } \mathcal{P} \text{ is a } \pi\text{-system then } \lambda(\mathcal{P}) \text{ is a } \sigma\text{-field}}$

Classic Application:

Ap #1 If $\mu_1(-\infty, x] = \mu_2(-\infty, x]$ for all x

then $\mu_1(A) = \mu_2(A)$ for all $A \in \sigma\{(-\infty, x] : x \in \mathbb{R}\} = \mathcal{B}$, Borel

Why: $\mathcal{P} = \{(-\infty, x] : x \in R\}$ is a π-system

$\mathcal{H} = \{A : \mu_1(A) = \mu_2(A)\}$ is a λ-system that contains $\mathcal{P}$

minimality. Hence $\mathcal{H}$ contains $\lambda(\mathcal{P})$ by minimality

but $\lambda(\mathcal{P})$ contains $\sigma(\mathcal{P})$ by minimality

so $\mathcal{H}$ contain $\sigma(\mathcal{P}) = \mathcal{B}$.

Ap #2 $\ldots \subset \mathcal{F}_n \subset \mathcal{F}_{n+1} \ldots$ Let $\mathcal{U} = \bigcup_{n=1}^{\infty} \mathcal{F}_n$, $(\Omega, \mathcal{F}, P)$

union π-system

Note: $\mathcal{U}$ need not be a σ-field (e.g. $\mathcal{F}_n$ binary fields)

But: $\mathcal{U}$ is a π-system.

approximation λ-system

$\boxed{\text{Let } \mathcal{H} = \{A \in \sigma(\mathcal{U}) : \forall \varepsilon > 0 \ \exists A_\varepsilon \in \mathcal{U} \text{ s.t. } P(A \,\Delta\, A_\varepsilon) \le \varepsilon\}}$

Note: (1) $\mathcal{U} \subset \mathcal{H}$ e.g. take $A_\varepsilon = A \in \mathcal{U}$.

(2) Prove H.W.: $\mathcal{H}$ is a λ-system

By Dynkin: $\sigma(\bigcup \mathcal{F}_n) \subset \mathcal{H}$. This is an approximation theorem

PROOF of Dynkin's Theorem

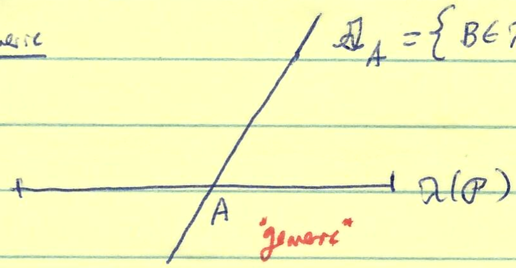
"Draw the "Same" Picture Three Times"

"good" for A

Generic

$\mathcal{D}_A = \{ B \in \lambda(\mathcal{P}) : A \cap B \in \lambda(\mathcal{P}) \}$

Same set can appear in two locations in the picture



$\boxed{\text{Inference: } \mathcal{D}_A \text{ is a } \lambda\text{-system}}$

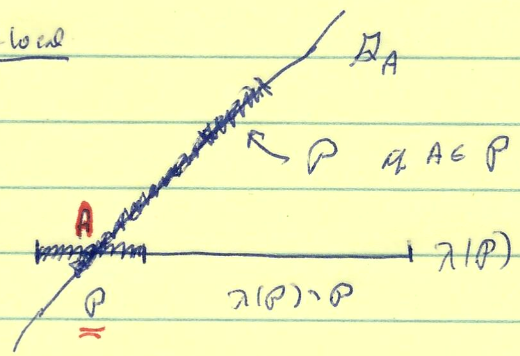
(i) ✓ since $A \in \lambda(\mathcal{P})$

(ii) $\boxed{\begin{matrix} A \cap B_1 \\ A \cap B_2 \end{matrix} \Rightarrow A \cap \{B_1 \setminus B_2\} = A \cap B_1 \setminus A \cap B_2}$

P-local

Here we use the hypothesis

Double repeated λ or



• Since $\mathcal{P}$ is a π-system, and $\mathcal{P} \subset \lambda(\mathcal{P})$

(A) • $\boxed{\text{Inference: } \lambda(\mathcal{P}) \subset \mathcal{D}_A}$

since $\mathcal{D}_A$ is a λ-system

Generic post symmetry

λ(P) P

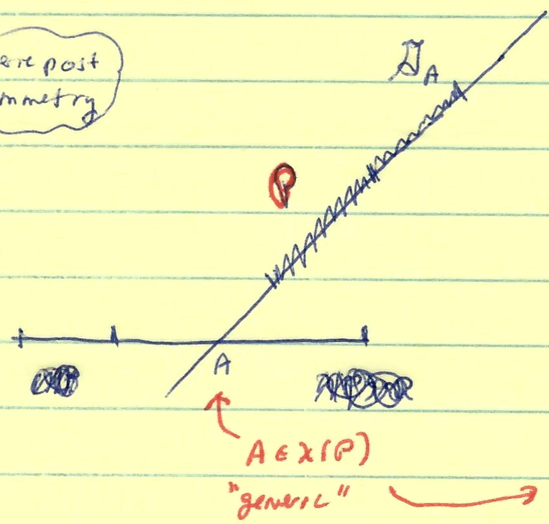
↓ ↙

{A, B} all good

{A, B} all good

↗ ↑

P λ(P)



"Symmetry"

(A) ⇒ $A \in \mathcal{P},\ B \in \lambda(\mathcal{P}) \Rightarrow A \cap B \in \lambda(\mathcal{P})$

So $A \in \lambda(\mathcal{P}),\ B \in \mathcal{P} \Rightarrow A \cap B \in \lambda(\mathcal{P})$

so for all $A \in \lambda(\mathcal{P})$

$\mathcal{D}_A$ contains $\mathcal{P}$

But $\mathcal{D}_A$ is a λ-system so

$\mathcal{D}_A$ contains $\lambda(\mathcal{P})$

Hence: For any A, B in $\lambda(\mathcal{P})$
$A \cap B \in \lambda(B)$ i.e. $\lambda(\mathcal{P})$ is a π-system.