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Some Tauberian Theorems and the Asymptotic Behavior of Probabilities of Recurrent Events

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INTRODUCTION

The problem we shall be concerned with here has been suggested by the theory of probability but can be formulated and treated in a purely analytical fashion.

We are given a sequence $\{f_n\}$ of real numbers satisfying the requirements

$$\begin{cases} f_n \geq 0, \qquad \sum_{n=1}^{\infty} f_n = 1 \\ (\text{greatest common divisor of the } n\text{'s such that } f_n > 0) = 1, \end{cases} \tag{I.1}$$

and we define a sequence $\{u_n\}$ by the equations

$$\begin{cases} u_0 = 1 \\ u_n = \sum_{k=1}^{n} f_k u_{n-k}, \qquad n \geq 1. \end{cases} \tag{I.2}$$

It is easy to see that for each n, $0 \leq u_n \leq 1$, and it is well known [9] that

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \left(\sum_{k=1}^{n} kf_k\right)^{-1}. \tag{I.3}$$

Our attention here will be devoted to the cases in which

$$\sum_{k=1}^{\infty} kf_k = \infty. \tag{I.4}$$

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This condition, in view of (I.3), implies that $u_n \to 0$ as $n \to \infty$. However, if no further assumptions about the f_n's are made, the behavior of $\{u_n\}$ may be very irregular. In general (see [3]) it is not even true that

$$u_{n+1} \sim u_n . \tag{I.5}$$

Nevertheless, Erdos and de Bruijn [2-4] established (1.5) when

$$\lim_{n\to\infty} (f_{n+1}/f_n) = 1,$$

and in some other interesting cases. They conjectured that perhaps (1.5) could be obtained under very general conditions upon the f_n's. More recently Orey [5] has shown that a result such as (1.5) has applications to the theory of Markov chains. This development brought again attention to the problem originally investigated by Erdos and de Bruijn. In a recent work [6] the result (1.5) has been established under the condition

$$\limsup_{n\to\infty} (f_{n+1}/f_n) \leq 1^1 \tag{I.6}$$

or even, less restrictively, under the condition

$$\limsup_{n\to\infty} \frac{f_{n+1} + f_{n+2} + \cdots + f_{n+N}}{f_n + f_{n-1} + \cdots + f_{n+N-1}} \leq 1 \qquad \text{(for some } N \geq 1). \tag{I.7}$$

We should also bring the attention to another work related to the present one. In [7], under different types of assumptions, some very precise results concerning the behavior of $\{u_n\}$ were obtained. Namely, under the condition

$$R_n = f_{n+1} + f_{n+2} + \cdots \sim c/n^\alpha \tag{I.8}$$

for some $\frac{1}{2} < \alpha < 1$ it has been established that

$$u_n \sim \frac{1}{cn^{1-\alpha}} \frac{\sin \pi\alpha}{\pi} . \tag{I.9}$$

Similar results have been found when the constant c is replaced by a slowly varying function. It is perhaps worth mentioning that (1.8) does not, in general, imply (1.9) when $0 < \alpha < \frac{1}{2}$. Nevertheless, when (1.8) holds, in any case it can be shown [7] that one has at least

$$\liminf_{n\to\infty} n^{1-\alpha} u_n \geq \frac{1}{c} \frac{\sin \pi\alpha}{\pi} . \tag{I.10}$$

[1] Actually (I.4) can occur only in the case of equality.

In the present paper we shall establish (1.5) under very general conditions. Although our results here include all the above mentioned results as special cases they are not best possible. The only necessary and sufficient conditions for (1.5) to hold, known to this date, are conditions involving the sequences $\{f_n\}$ and $\{u_n\}$ simultaneously (see [3], [6], and Theorems 1.42 and 2.3 of the present paper) and cannot be considered satisfactory.

Perhaps the two main corollaries of our results here are the following theorems.

THEOREM I.1. *If conditions (I.1) are satisfied and in addition for some* $\alpha > 1$ *we have*

$$\limsup_{n\to\infty} \frac{\alpha f_1 + \cdots + \alpha^{n+1} f_{n+1}}{\alpha f_1 + \cdots + \alpha^n f_n} \leq \alpha,$$

then

$$u_{n+1} \sim u_n .$$

To state the next theorem we need to introduce an auxiliary sequence $\{\alpha_n\}$. For large n we let α_n be the positive solution of the equation

$$f_1\alpha_n + f_2\alpha_n^2 + \cdots + f_n\alpha_n^n = 1. \tag{I.11}$$

This defines α_n for $n \geq n_0$ where f_{n_0} is the first $f_n > 0$. For $n \leq n_0 - 1$ it is convenient to set

$$\alpha_n = \alpha_{n_0} .$$

It is easy to see that $\{\alpha_n\}$ is a nonincreasing sequence of numbers approaching one.

THEOREM I.2. *If the conditions (I.1) are satisfied and in addition the series*

$$\sum_n f_n \,\alpha_1\alpha_2 \cdots \alpha_n \tag{I.12}$$

is convergent, then

$$u_{n+1} \sim u_n .$$

Theorem I.1 was conjectured by S. Orey. It has the advantage over the previous results in that it allows the sequence f_n to have arbitrarily large gaps, while the only known counterexamples to (1.5) have been obtained by introducing such gaps.

From Theorem I.2 it is readily deduced that the condition

$$R_n = O[1/n]$$

implies (I.5). The following corollary of Theorem I.2 is also worth noting

THEOREM I.3. *If the sequence f_n satisfies (I.1) and the sequence $R_n = f_{n+1} + f_{n+2} + \cdots$ is such that for some α in the range $(\sqrt{5} - 1)/2, 1)$ we have*

A. $R_n = 0[1/n^\alpha]$.

B. $\liminf_{n\to\infty} (R[n\sigma]/R_n) \geq 1/\sigma^\alpha \qquad$ (for all $0 < \sigma < 1$).

then

$$u_{n+1} \sim u_n .$$

We should mention that the constant $(\sqrt{5} - 1)/2$ in Theorem I.3 is not the best possible. The result there can be improved by replacing $(\sqrt{5} - 1)/2$ by the number $\alpha_0 > \frac{1}{2}$ defined by the equation

$$e^{\alpha_0} = \alpha_0 \int_0^1 e^{\alpha_0 \sigma} \, d\sigma / \sigma^{\alpha_0} . \tag{I.13}$$

Perhaps the best constant in Theorem I.3 is $\frac{1}{2}$. It is also a conjecture whether or not the assumption $R_n \sim c/n^\alpha$ implies (I.5) also for $0 < \alpha \leq \frac{1}{2}$.

I. NOTATIONS AND AUXILIARY RESULTS

1.1 For convenience we shall introduce the generating functions

$$F(t) = \sum_{n=1}^{\infty} f_n t^n, \qquad R(t) = \sum_{n=0}^{\infty} R_n t^n, \qquad U(t) = \sum_{n=0}^{\infty} u_n t^n. \tag{1.11}$$

The following relations hold.

$$U(t) = \frac{1}{1 - F(t)} = \frac{1}{(1 - t) R(t)} . \tag{1.12}$$

We shall also set

$$u_{n+1}/u_n = r_n .$$

A very convenient method of establishing Tauberian theorems is one that is essentially due to Beurling [8]. We shall introduce it in the form needed in the present context. Suppose we are in possession of a bound of the form

$$\limsup_{n\to\infty} r_n \leq M < \infty.$$

Then by a diagonal process we can find a sequence n_k such that all variables

$$r_{n_k+l} \qquad (l = 0, \pm 1, \pm 2, \cdots)$$

are convergent to finite limits.

We set

$$\lim_{k\to\infty} r_{n_k+l} = \sigma_l . \tag{1.13}$$

The idea is that the sequence n_k can be chosen so that σ_0 results equal to a particular limit of the sequence r_n. For instance we can make so that

$$\sigma_0 = \liminf_{n\to\infty} r_n \qquad \text{or} \qquad \sigma_0 = \limsup_{n\to\infty} r_n .$$

We then proceed to find relations between the numbers σ_l which eventually imply estimates upon σ_0.

This approach was used with success, in this problem, in [6]. To simplify our exposition a subsequence as described above will be referred to as "a determining sequence."

1.2 We shall recall a few results which will be of use in the following. First of all from (I.1) and (I.2) it can be easily shown that $u_n > 0$ for all sufficiently large n.

We also have the inequality

$$u_n \geq u_k u_{n-k} \qquad (\text{for all } 0 \leq k \leq n) \tag{1.21}$$

This yields in particular that for all sufficiently large N

$$u_n = O[u_{n+N}]. \tag{1.22}$$

We can then deduce

LEMMA 1.2. *A necessary and sufficient condition for*

$$u_{n+1} = O[u_n] \tag{1.23}$$

is that there exists an $N_0 \geq 1$ *such that*

$$u_n = O[u_{n-1} + u_{n-2} + \cdots + u_{n-N_0}]. \tag{1.24}$$

PROOF. The necessity is obvious. As for the sufficiency we note that if (1.22) is true for all $N \geq N_1$, then in view of (1.24) we have

$$u_n = O[u_{n+N_1-1} + u_{n+N_1-1} + \cdots + u_{n+N_1-1}] = O[u_{n+N_1-1}].$$

This implies the validity of (1.22) for all N, in particular (1.23). The lemma gives a useful criterion.

CRITERION. *If there exists an N such that*

$$f_n = O[f_{n-1} + f_{n-2} + \cdots + f_{n-N}] \tag{1.25}$$

then

$$\limsup_{n\to\infty} (u_{n+1}/u_n) < \infty.$$

Such a result was noticed and used in both [3] and [6]. It can be established by showing that (1.25) implies (1.24).

1.3 If $f_1 = 0$, $\liminf_{n\to\infty} r_n$ need not be different from zero but if

$$\limsup_{n\to\infty} r_n = M < \infty \tag{1.31}$$

then we have

$$\liminf_{n\to\infty} r_n \geq MF(1/M). \tag{1.32}$$

In view of the definition (1.11) of $F(t)$ and (I.1) we get

THEOREM 1.3. *When (I.1), (I.2) hold, in order that*

$$u_{n+1} \sim u_n$$

it is necessary and sufficient that

$$\limsup_{n\to\infty} (u_{n+1}/u_n) \leq 1. \tag{1.33}$$

1.4 Let us assume that (1.31) holds and let n_k be an arbitrary determining sequence. From (I.2) we deduce that for $n > N$

$$u_n \geq \sum_{k=1}^{N} f_k u_{n-k}$$

dividing by u_n

$$1 \geq \sum_{k=1}^{N} \frac{f_k}{r_{n-1} \cdots r_{n-k}},$$

passing to the limit along $n = n_k + l$ first, then letting $N \to \infty$ and using (1.13) we obtain

$$1 \geq \sum_{k=1}^{\infty} \frac{f_k}{\sigma_{l-1} \cdots \sigma_{l-k}} \qquad (l = 0, \pm 1, \pm 2, \cdots). \tag{1.41}$$

Note that these relations are equivalent to

$$\sigma_l \geq \sum_{k=1}^{\infty} \frac{f_k}{\sigma_{l-1} \cdots \sigma_{l-k}} \sigma_{l-k} \qquad (l = 0, \pm 1, \pm 2, \cdots). \tag{1.42}$$

Thus, in particular, the inequality (1.32) is obtained when

$$\sigma_0 = \liminf_{n\to\infty} r_n .$$

We note that equality in (1.41) cannot be assured for all determining sequences and for all l without establishing that $u_{n+1} \sim u_n$. As a matter of fact we have

THEOREM 1.41. *For all determining sequences we have*

$$\sum_{k=1}^{\infty} \frac{f_k}{\sigma_{l-1} \cdots \sigma_{l-k}} = 1 \qquad (l = 0, \pm 1, \pm 2, \cdots) \tag{1.43}$$

if and only if

$$u_{n+1} \sim u_n .$$

A proof of this result can be found in [6].

We shall also recall that

THEOREM 1.42. *The equalities in (1.43) and therefore $u_{n+1} \sim u_n$ hold if and only if*

$$\lim_{N\to\infty} \limsup_{n\to\infty} \frac{1}{u_n} \Big(\sum_{k=N+1}^{n} f_k u_{n-k} \Big) = 0. \tag{1.44}$$

A similar but slightly more complicated necessary and sufficient condition can be found in [3].

From the inequality (1.21) and Theorem 1.42 we easily deduce the

CRITERION 1.4. *A sufficient condition for $u_{n+1} \sim u_n$ is that the series*

$$\sum_n f_n / u_n \tag{1.45}$$

is convergent.

This criterion was also mentioned in [3].

II. Proofs of the Main Results

2.1 The criterion 1.4 suggests looking for some suitable lower bounds for the sequence u_n. The importance of such bounds was already noticed in [1]. The results there (cf. Theorem 2.2, p. 4) are of the form

$$u_n \geq (1-\epsilon)^n \qquad \text{for all} \qquad n \geq n(\epsilon) \tag{2.11}$$

and for every $\epsilon > 0$. Such bounds can be obtained immediately from (I.3) In fact, observe that for large N the sequence

$$f_1' = f_1\alpha_N\,, \quad f_2' = f_2\alpha_N^2\,, \quad \cdots, \quad f_N' = f_N\alpha_N^N; \qquad f_n' = 0 \quad \text{for} \quad n > N$$

satisfies I.1, so that the sequence u_n' defined by (I.2), in view of (I.3), satisfies

$$\lim_{n\to\infty} u_n' = 1\Big/\sum_{k=1}^{N} k\alpha_N^k\,.$$

On the other hand it is clear that we have

$$u_n \geq u_n'/\alpha_N^n \qquad \text{for all} \qquad n \geq 0. \tag{2.12}$$

It was also shown in [1] that (2.11) cannot in general be further improved without taking into account the behavior of the f_n's. Nevertheless, in each particular case, (2.11) may be a long way from reflecting the behavior of the u_n's. A more satisfactory type of bound is the one given by the following

Theorem 2.1. *If the f_n's satisfy (I.1) and the α_n's are defined by (I.11), then there exist a constant $A > 0$ and an integer n_0 such that*

$$u_{n_0+k} \geq A/(\alpha_1\alpha_2 \cdots \alpha_k) \qquad \text{for all} \qquad k \geq 0. \tag{2.13}$$

Proof. For simplicity we shall prove the theorem under the assumption that $f_1 > 0$. We shall then have $u_n > 0$ for all n. Thus there exists a constant A so that

$$u_n \geq A/(\alpha_1\alpha_2 \cdots \alpha_n) \tag{2.14}$$

at least for $n = 0, 1$. We proceed by induction and assume (2.14) true for $0 \leq n \leq m-1$. We then have

$$\alpha_1\alpha_2 \cdots \alpha_m u_m = \sum_{k=1}^{m} f_k[\alpha_1\alpha_2 \cdots \alpha_{m-k}u_{m-k}]\, \alpha_{m-k+1} \cdots \alpha_m\,.$$

Using the definition of $\{\alpha_n\}$ and (2.14)

$$\alpha_1\alpha_2 \cdots \alpha_m u_m \geq A \sum_{k=1}^{m} f_k \alpha_m^k = A.$$

This proves the theorem.

Combining the estimate (2.14) with criterion 1.4 we obtain a proof of Theorem I.2 stated in the introduction.

2.2 To prove Theorem I.3 we set

$$G(x) = \sum_{k \leqslant x} f_k \,, \qquad R(x) = 1 - G(x), \qquad \alpha_n = e^{A_n/n}.$$

The definition of α_n gives that

$$1 = \int_0^n e^{(A_n/n)x}\, dG(x) = -\int_0^n e^{(A_n/n)x}\, d[1 - G(x)]. \tag{2.21}$$

And we obtain

$$1 \geq \int_0^n \left(1 + \frac{A_n}{n}\, x\right) dG(x) = 1 - R_n - \frac{A_n}{n} \int_0^n x\, dR(x). \tag{2.22}$$

Integrating by parts and making the substitution $x = n\sigma$ in both (2.21) and (2.22) we get

$$A_n \leq \frac{nR_n}{n \int_0^1 R(n\sigma)\, d\sigma - nR_n} = \frac{1}{\int_0^1 [R(n\sigma)/R(n)]\, d\sigma - 1} \tag{2.23}$$

$$e^{A_n} = A_n \int_0^1 e^{A_n\sigma} \frac{R(n\sigma)}{R(n)}\, d\sigma. \tag{2.24}$$

Under the assumption B of Theorem I.3 we get (by Fatou's lemma)

$$A = \limsup_{n\to\infty} A_n \leq \frac{1-\alpha}{\alpha}. \tag{2.25}$$

Passing to the limit in (2.24) along a suitable subsequence and using Fatou's lemma again, from (2.24) we obtain

$$e^A \geq A \int_0^1 e^{A\sigma}\, (d\sigma/\sigma^\alpha). \tag{2.26}$$

This inequality implies that

$$A \leq A(\alpha) \tag{2.27}$$

where $A(\alpha)$ is the number A which reduces (2.26) to an equality.

From (2.27) we get

$$\alpha_1\alpha_2 \cdots \alpha_n = O[n^{A(\alpha)+\epsilon}] \qquad \text{for any} \qquad \epsilon > 0.$$

Using assumption A of Theorem I.3 we obtain that the series

$$\sum f_n \, \alpha_1\alpha_2 \cdots \alpha_n$$

is convergent as soon as the series

$$\sum \frac{n^{A(\alpha)+\epsilon}}{n^{1+\alpha}}$$

is convergent. That is, when

$$A(\alpha) < \alpha.$$

But this occurs when

$$\alpha > \alpha_0 ,$$

α_0 being the solution of Eq. (I.13).

Remark. We should mention that when

$$R_n = O[1/n] \tag{2.28}$$

the inequality (2.22) (together with the assumption (I.4)) gives

$$A_n/n = o[R_k] = o[1/n]$$

and then also

$$\alpha_1\alpha_2 \cdots \alpha_n = o\,[\log n].$$

This result combined with (2.28) yields the convergence of the series (I.12).

2.3. We shall obtain the proof of Theorem I.1 after several steps. We start by establishing:

THEOREM 2.3. *A necessary and sufficient condition for $u_{n+1} \sim u_n$ is that there exists a sequence of nonnegative numbers $\{\pi_\nu\}$ (not all vanishing) and an integer $\rho \geq 0$ such that*

$$\lim_{N\to\infty} \limsup_{n\to\infty} \frac{1}{u_{n+\rho}} \sum_{k=N+1}^{n} \pi_k u_{n-k} = 0, \tag{2.31}$$

$$\limsup_{n\to\infty} \frac{\sum_{\nu=0}^{n+1} \pi_\nu u_{n+1-\nu}}{\sum_{\nu=0}^{n} \pi_\nu u_{n-\nu}} \leq 1. \tag{2.32}$$

PROOF. The necessity is quite clear. (When $\pi_0 = 1$ and $\pi_n = 0$, $n > 0$ it is trivial; when $\pi_0 = 0$ and $\pi_n = f_n$, $n > 0$ it follows from Theorem 1.42.)

The sufficiency is more difficult and will require two auxiliary lemmas. We first observe that since ρ may be any integer greater than zero, there is no loss of generality in assuming that $\pi_0 > 0$.

LEMMA 2.31. *Under the hypotheses (2.31) and (2.32) we have*

$$M = \limsup_{n\to\infty} u_{n+1}/u_n < \infty. \tag{2.33}$$

PROOF. From (2.32) we have

$$\sum_{\nu=0}^{n+1} \pi_\nu u_{n+1-\nu} = O\left[\sum_{\nu=0}^{n-\rho} \pi_\nu u_{n-\rho-\nu}\right]. \tag{2.34}$$

But (2.31) implies that for N large enough we have

$$\sum_{\nu=N+1}^{n} \pi_\nu u_{n-\nu} = O[u_{n+\rho}]. \tag{2.35}$$

Combining (2.35) with (2.34) we get

$$\pi_0 u_{n+1} = O\left[\sum_{\nu=0}^{N} \pi_\nu u_{n-\rho-\nu}\right] + O[u_n],$$

in other words

$$u_{n+1} = O[u_n + u_{n-1} + \cdots + u_{n-N-\rho}].$$

Thus Lemma 2.31 follows from Lemma 1.2.

LEMMA 2.32. *Under the hypotheses (2.31) and*

$$M = \limsup_{n\to\infty} u_{n+1}/u_n < \infty \tag{2.36}$$

there exists a constant Γ such that for every determining sequence and for every l

$$\Gamma_l = \sum_{\nu=0}^{\infty} \frac{\pi_\nu}{\sigma_{l-1} \cdots \sigma_{l-\nu}} \le \Gamma. \tag{2.37}$$

In addition for every determining sequence $\{n_k\}$ we have

$$\lim_{k\to\infty} \frac{1}{u_{n_k+l}} \sum_{\nu=0}^{n_k+l} \pi_\nu u_{n_k+l-\nu} = \sum_{\nu=0}^{\infty} \frac{\pi_\nu}{\sigma_{l-1} \cdots \sigma_{l-\nu}}. \tag{2.38}$$

PROOF. Let $\{n_k\}$ be a determining sequence. From (2.31) and (2.36) we deduce that for a given $\epsilon > 0$ and a sufficiently large N_ϵ

$$\frac{1}{u_n}\sum_{\nu=N_\epsilon+1}^{n} \pi_\nu u_{n-\nu} \leq \epsilon \qquad \text{for all} \qquad n \geq n(N_\epsilon).$$

Let now N be arbitrary and $m = \lim_{n\to\infty} \inf r_n$ (in view of (2.36) $m > 0$). We shall have

$$\sum_{\nu=0}^{N} \frac{\pi_\nu}{r_{n-1}\cdots r_{n-\nu}} \leq \sum_{\nu=0}^{n} \frac{\pi_\nu}{r_{n-1}\cdots r_{n-\nu}} \leq \sum_{\nu=0}^{N_\epsilon} \frac{\pi_\nu}{r_{n-1}\cdots r_{n-\nu}} + \epsilon. \tag{2.39}$$

Passing to the limit along $n = n_k + l$ we get

$$\sum_{\nu=0}^{N} \frac{\pi_\nu}{\sigma_{l-1}\cdots\sigma_{l-\nu}} \leq \sum_{\nu=0}^{N_\epsilon} \frac{\pi_\nu}{m^\nu} + \epsilon.$$

Since N is arbitrary, this inequality implies the first statement of the lemma. The remaining part of the lemma is obtained by first passing to the limit in (2.39) along $n = n_k + l$ and then letting N and N_ϵ tend to infinity.

2.4 We are now in a position to complete the proof of Theorem 2.3. We shall achieve this by showing that for every determining sequence $\{n_k\}$ and every l we have

$$\sum_{k=1}^{\infty} \frac{f_k}{\sigma_{l-1}\cdots\sigma_{l-k}} = 1. \tag{2.41}$$

In other words we shall reduce Theorem 2.3 to Theorem 1.41. Formula (2.41) can be established as follows. The assumption (2.32) in view of (2.38) yields

$$\sigma_l\,(\Gamma_{l+1}/\Gamma_l) \leq 1. \tag{2.42}$$

By a repeated application of this inequality we obtain that for every $k \geq 1$

$$\Gamma_l \leq \frac{\Gamma_{l-k}}{\sigma_{l-1}\cdots\sigma_{l-k}}\,.$$

Multiplying by f_k and summing

$$\Gamma_l \leq \sum_{k=1}^{\infty} \frac{f_k}{\sigma_{l-1}\cdots\sigma_{l-k}}\,\Gamma_{l-k}\,. \tag{2.43}$$

We note that the series on the right hand side of this inequality is convergent because of (1.41) and the uniform boundedness of the Γ_l's (2.37.) Since the terms of this series are nonnegative, we can sum them in any order we please. For instance, using (2.37)

$$\sum_{k=1}^{\infty} \frac{f_k \Gamma_{l-k}}{\sigma_{l-1} \cdots \sigma_{l-k}} = \sum_{k=1}^{\infty} f_k \sum_{\nu=0}^{\infty} \frac{\pi_\nu}{\sigma_{l-1} \cdots \sigma_{l-k-\nu}},$$

and using (1.41)

$$\sum_{k=1}^{\infty} \frac{f_k \Gamma_{l-k}}{\sigma_{l-1} \cdots \sigma_{l-k}} = \sum_{\nu=0}^{\infty} \frac{\pi_\nu}{\sigma_{l-1} \cdots \sigma_{l-\nu}} \sum_{k=1}^{\infty} \frac{f_k}{\sigma_{l-\nu-1} \cdots \sigma_{l-\nu-k}} \leq \Gamma_l .$$

This inequality reverses (2.43). But this can be possible only if

$$\sum_{k=1}^{\infty} \frac{f_k}{\sigma_{l-\nu-1} \cdots \sigma_{l-\nu-k}} = 1$$

for each ν such that $\pi_\nu > 0$. Since $\pi_0 > 0$ and l is arbitrary, we obtain 2.41.

2.5 Theorem I.1 is a corollary of the following:

THEOREM 2.5. *If the sequence f_n satisfies (I.1) and in addition there exists a nonnegative sequence $\{\pi_n\}$ and an integer $\rho \geq 0$ such that*

$$\lim_{N\to\infty} \limsup_{n\to\infty} \left[\frac{1}{u_{n+\rho}} \sum_{\nu=N+1}^{n} \pi_\nu u_{n-\nu}\right] = 0 \tag{2.51}$$

$$\limsup \frac{\pi_n f_1 + \pi_{n-1} f_2 + \cdots + \pi_0 f_{n+1}}{\pi_{n-1} f_1 + \pi_{n-2} f_2 + \cdots + \pi_0 f_n} \leq 1 \tag{2.52}$$

then $u_{n+1} \sim u_n$.

PROOF. Here again since ρ may be greater than zero, we can assume without loss that $\pi_0 > 0$. For convenience we introduce the constants

$$A_n = \pi_{n-1} f_1 + \pi_{n-2} f_2 + \cdots + \pi_0 f_n \tag{2.53}$$

and the functions

$$A(t) = \sum_{n=1}^{\infty} A_n t^n, \qquad \pi(t) = \sum_{\nu=0}^{\infty} \pi_\nu t^\nu.$$

We then have $A(t) = \pi(t) F(t)$ so that using the formulas (1.12) we obtain

$$\pi(U - 1) = UA.$$

Equating coefficients we get

$$\sum_{\nu=0}^{n-1} \pi_\nu u_{n-\nu} = \sum_{\nu=1}^{n} A_\nu u_{n-\nu} \,. \tag{2.54}$$

From (2.52) we get that for any given $\lambda > 1$ we can find $n(\lambda)$ so that $A_{\nu+1} \leq \lambda A_\nu$ for all $\nu \geq n(\lambda)$. Using this fact in (2.54) for $n > N > n(\lambda) + \rho + 1$ we obtain

$$\sum_{\nu=0}^{n-1} \pi_\nu u_{n-\nu} \leq \sum_{\nu=1}^{N} A_\nu u_{n-\nu} + \lambda^{\rho+1} \sum_{\nu=N+1}^{n} A_{\nu-\rho-1} u_{n-\nu}$$

$$= \sum_{\nu=1}^{N} A_\nu u_{n-\nu} + \lambda^{\rho+1} \sum_{\nu=N-\rho}^{n-\rho-1} A_\nu u_{n-\rho-1-\nu} \,.$$

Using (2.54) again we get

$$\sum_{\nu=0}^{n-1} \pi_\nu u_{n-\nu} \leq \sum_{\nu=1}^{N} A_\nu u_{n-\nu} + \lambda^{\rho+1} \left[\sum_{\nu=0}^{n-\rho-2} \pi_\nu u_{n-\rho-1-\nu} - \sum_{\nu=0}^{N-\rho-1} A_\nu u_{n-\rho-1-\nu} \right] . \tag{2.55}$$

From (2.51) we get that

$$\sum_{\nu=N-\rho}^{n-\rho-2} \pi_\nu u_{n-\rho-1-\nu} = O[u_{n-1}].$$

Combining this estimate with (2.55) we obtain that

$$u_n = O[u_{n-1} + u_{n-2} + \cdots + u_{n-N}].$$

Thus, in view of Lemma 1.2 we get $u_{n+1} = 0[u_n]$. This result implies that (2.51) must also be true with $\rho = 0$.

2.6 We shall reduce Theorem 2.5 to Theorem 2.3 by showing that

LEMMA 2.6. *Under the assumption (2.51), if for some $\Lambda \geq 1$, we have*

$$\limsup_{n\to\infty} \frac{\pi_n f_1 + \pi_{n-1} f_2 + \cdots + \pi_0 f_{n+1}}{\pi_{n-1} f_1 + \pi_{n-2} f_2 + \cdots + \pi_0 f_n} \leq \Lambda \tag{2.61}$$

then we must also have

$$\Lambda' = \limsup_{n\to\infty} \frac{\pi_{n=1} u_0 + \pi_n u_1 + \cdots + \pi_0 u_{n=1}}{\pi_n u_0 + \pi_{n-1} u_1 + \cdots + \pi_0 u_n} \leq \Lambda. \tag{2.62}$$

PROOF. In Section 2.5 we have essentially shown that (2.51) and (2.61) imply

$$M = \limsup_{n\to\infty} \frac{u_{n+1}}{u_n} < \infty. \tag{2.63}$$

We can thus choose a determining sequence $\{n_k\}$ such that

$$\lim_{k\to\infty} \frac{\pi_{n_k+1}u_0 + \pi_{n_k}u_1 + \cdots + \pi_0 u_{n_k+1}}{\pi_{n_k}u_0 + \pi_{n_k-1}u_1 + \cdots + \pi_0 u_{n_k}} = \Lambda'. \tag{2.64}$$

Since we are assuming (2.51) and we have (2.63), Lemma 2.32 applies. From it, (2.64), and the definition of Λ' we obtain that

$$\frac{\sigma_l \Gamma_{l+1}}{\Gamma_l} \le \Lambda', \qquad \frac{\sigma_0 \Gamma_1}{\Gamma_0} = \Lambda'. \tag{2.65}$$

This also gives that $\Lambda' \le M\Gamma/\pi_0 < \infty$.

We note that under the assumption of this lemma the inequality (2.55) must hold for any $\lambda > \Lambda$ and for $\rho = 0$. Dividing (2.55) by u_n and passing to the limit along $n = n_k + l$ we obtain

$$\Gamma_l \le \sum_{\nu=1}^{N} \frac{A_\nu}{\sigma_{l-1}\cdots\sigma_{l-\nu}} + \frac{\lambda}{\sigma_{l-1}}\left[\Gamma_{l-1} - \sum_{\nu=0}^{N-1} \frac{A_\nu}{\sigma_{l-2}\cdots\sigma_{l-\nu-1}}\right]. \tag{2.66}$$

Since N may be arbitrarily large, we shall pass to the limit as $N \to \infty$. But before doing so we observe that by (2.53), (2.37), and (1.41) we have

$$\sum_{\nu=1}^{\infty} \frac{A_\nu}{\sigma_{l-1}\cdots\sigma_{l-\nu}} = \sum_{k=1}^{\infty} \frac{f_k \Gamma_{l-k}}{\sigma_{l-1}\cdots\sigma_{l-k}}$$

$$= \sum_{\nu=0}^{\infty} \frac{\pi_\nu}{\sigma_{l-1}\cdots\sigma_{l-\nu}} \sum_{k=1}^{\infty} \frac{f_k}{\sigma_{l-\nu-1}\cdots\sigma_{l-\nu-k}} \le \Gamma_l < \infty. \tag{2.67}$$

Thus (2.66) yields

$$\Gamma_l \le \sum_{k=1}^{\infty} \frac{f_k}{\sigma_{l-1}\cdots\sigma_{l-k}} \Gamma_{l-k} + \frac{\lambda}{\sigma_{l-1}}\left[\Gamma_{l-1} - \sum_{k=1}^{\infty} \frac{f_k\,\Gamma_{l-k-1}}{\sigma_{l-2}\cdots\sigma_{l-k-1}}\right]. \tag{2.68}$$

Since λ may be an arbitrary number greater than Λ, we can replace λ by Λ in (2.68). The resulting inequality is best written in the form

$$\left(\Gamma_l - \Lambda \frac{\Gamma_{l-1}}{\sigma_{l-1}}\right) \le \sum_{k=1}^{\infty} \frac{f_k}{\sigma_{l-1}\cdots\sigma_{l-k}} \left(\Gamma_{l-k} - \Lambda \frac{\Gamma_{l-k-1}}{\sigma_{l-k-1}}\right). \tag{2.69}$$

2.7 Let us now assume, if possible, that

$$\Lambda' > \Lambda. \tag{2.71}$$

Note that for $l = 1$ we have (in view of (2.65))

$$\Gamma_l - \Lambda \frac{\Gamma_{l-1}}{\sigma_{l-1}} = \Gamma_l \left(1 - \frac{\Lambda}{\Lambda'}\right). \tag{2.72}$$

Suppose then that (2.72) holds for a given l. From (2.69), using (2.65), (2.67), and (2.71), we get

$$\Gamma_l \left(1 - \frac{\Lambda}{\Lambda'}\right) \leq \sum_{k=1}^{\infty} \frac{f_k \Gamma_{l-k}}{\sigma_{l-1} \cdots \sigma_{l-k}} \left(1 - \frac{\Lambda}{\Lambda'}\right) \leq \Gamma_l \left(1 - \frac{\Lambda}{\Lambda'}\right). \tag{2.73}$$

This implies not only that (2.73) must be an equality, but also that

$$\frac{\Gamma_{l-k}}{\Lambda'} = \frac{\Gamma_{l-k-1}}{\sigma_{l-k-1}} \tag{2.74}$$

at least for all k such that $f_k > 0$. However, then the aperiodicity condition in (I.1) assures the existence of an l such that (2.74) holds for every $k \geq 0$.

Making a repeated use of (2.74) we obtain

$$\frac{\Gamma_l}{(\Lambda')^k} = \frac{\Gamma_{l-k}}{\sigma_{l-1} \cdots \sigma_{l-k}}.$$

Multiplying this relation by f_k and summing, we obtain

$$\Gamma_l \sum_{k=1}^{\infty} \frac{f_k}{(\Lambda')^k} = \sum_{k=1}^{\infty} \frac{f_k \Gamma_{l-k}}{\sigma_{l-1} \cdots \sigma_{l-k}} \geq \Gamma_l,$$

but this is absurd if $\Lambda' > 1$.

Remarks. We shall close by showing how Theorem I.1 may be deduced from Theorem 2.5. Suppose that for some $\alpha > 1$

$$\limsup_{n \to \infty} \frac{\alpha f_1 + \alpha^2 f_2 + \cdots + \alpha^{n+1} f_{n=1}}{\alpha f_1 + \alpha^2 f_2 + \cdots + \alpha^n f_n} \leq \alpha.$$

This condition, setting

$$\pi_n = (1/\alpha)^n, \tag{2.75}$$

is easily seen to imply (2.52).

We are thus left to verify that condition (2.51) is always satisfied by a sequence such as (2.75). This, however, is an easy consequence of (1.21) and the lower bounds (2.11). In fact we have

$$\frac{1}{u_n} \sum_{k=N+1}^{n} \pi_k u_{n-k} \leq \sum_{k=N+1}^{\infty} \frac{1}{\alpha^k u_k} \leq \sum_{k=N+1}^{\infty} \frac{1}{[\alpha(1-\epsilon)]^k} < \infty$$

at least when ϵ is sufficiently small.

We should also mention that condition (2.51) is trivially satisfied when π_n is a sequence that has only a finite number of non vanishing terms. Thus from Theorem 2.5 we obtain also that the condition

$$\limsup_{n\to\infty} \frac{\pi_0 f_{n+1} + \cdots + \pi_N f_{n+1-N}}{\pi_0 f_n + \cdots + \pi_N f_{n-N}} \leq 1$$

for some nonnegative constants $\pi_0, \pi_1, \cdots, \pi_N$ is sufficient to guarantee that

$$u_{n+1} \sim u_n .$$

This result was announced without proof in [6].

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