Statistics 931: Probability Theory
Homework No. 2

INSTRUCTIONS. Make sure you understand our two proofs of "the" Choquet-Deny lemma, and make sure you understand the logic of the proof of the discrete renewal theorem using the Choquet-Deny lemma. The mnemonic GRaVy stands for "generalization, refinement, and variation." Any time you see a mathematical fact, it is useful to ask if it also comes with some GRaVy. This is one of the best tools I know for making yourself creatively engage the material that you are studying. Sometimes GRaVy is easy or obvious, sometimes it is simply invisible to our first look. It is a powerful "stand" to declare to yourself that it is always there.

For the first problem you should review the material on Wald's lemma in the text. You should also read opportunistically about renewal theory and Markov chain theory. In one way or another, we will cover most of the material in the text. For this week, you just have two problems, so DO take the reading seriously.

PROBLEM 1.[Wald Lemma GRaVy]. Suppose that X_n, $n = 1, 2, \ldots$ is an i.i.d. sequence of *vector valued* random variables with finite first moment. Let $\mu = EX_1$ and suppose that τ is a stopping time with finite mean. Explain why the real-valued Wald lemma implies

$$ES_\tau = \mu E\tau.$$

Now suppose $E||X_1||_2^2 < \infty$ and consider the possibility of a "Variance Wald Lemma" for vector valued random variables. As a first step, suppose that μ is zero and prove or disprove that

$$E[S_\tau S_\tau^T] = E\tau E[X_1 X_1^T].$$

Here v^T denotes the transpose of the column vector v so vv^T is an n by n matrix.

PROBLEM 2.[Probability and Partial Fractions]. Suppose independent random variables X_i, $i = 1, 2, \ldots$ satisfy $P(X_i = 1) = P(X_i = 2) = 1/2$. Let $f(s) = (s + s^2)/2$ so f is the probability generating function of X_1.

- Use first step analysis to find
$$U(s) = \sum_{k=1}^{\infty} p_k s^k \quad \text{where } p_k = P(S_n = k \quad \text{for some } n \geq 1).$$
- Use partial fractions to find a formula for p_k.
- Use your formula to find the limit of p_k as $k \to \infty$ and give a "gambler's" explanation of why your formula is right.

PHILOSOPHY. As you read always think about GRaVy. No result stands alone. If you have a result in front of you, also have some generic questions:

- Is there a simpler result — with a simpler, or at least different, proof?
- Is there a reasonable generalization? Complex valued random variables? Vector valued random variables?
- What have I really used? Maybe I assumed independence, but, if all I used was orthogonality, then I really have something much more general than initially claimed.

Make yourself a list of generic questions and give yourself some practice using them. Every once in a while, you'll turn up something good.