FELLER’S PROOF OF THE DISCRETE ARCSINE LAW
— THE ESSENTIAL STEPS

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ABSTRACT. We simplify and clarify Feller’s proof of the discrete arcsine law.

We consider the simple symmetric random walk, where $X_1, \ldots, X_n$ are independent Rademacher random variables and $S_n = X_1 + \cdots + X_n$ with $S_0 = 0$. It is illustrative to envision a sequence $\{S_k\}_{1 \leq k \leq n}$ such that $S_n = b$ as a path from 0 to b with vertices $S_k$, $1 \leq k \leq n$ for each time step $k$. In visualizing a random walk so, its symmetrical properties are immediately tangible. Following Feller’s work, we explore how these simple properties may be exploited to great effect, eventually yielding counter-intuitive results. We begin our discussion with the following symmetry result.

Lemma 1 (Reflection principle). Let $a$ and $b$ be positive integers. Then the number of paths with $S_m = a$ and $S_n = b$ that return to zero equals the number of paths with $S'_m = -a$ and $S'_n = b$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{reflection_principle.png}
\caption{Reflection principle. The dashed line shows how to construct the bijection between a path with $S_m = a$ and $S_n = b$ that return to zero and a path with $S'_m = -a$ and $S'_n = b$.}
\end{figure}

Proof. We proceed by proving an injection from the paths with $S_m = a$ and $S_n = b$ that return to zero into the paths with $S_m = -a$ and $S_n = b$, and then we prove an injection in the opposite direction to show the number of paths is equivalent.

Consider a path with $S_m = a, \ldots, S_n = b$ where $a, b > 0$ and let $T$ be the first time the path returns to zero (i.e. $T = \min\{t : S_t = 0\}$). Define $U_t$ to be

$$U_t = \begin{cases} -S_t & \text{for } t \leq T \\ S_t & \text{for } t > T. \end{cases}$$

Then $\{U_m, \ldots, U_n\}$ is a path from $-a$ to $b$. 

Now we prove the reverse injection. Suppose \( S'_m = -a, \ldots, S'_n = b \) is a path. Let \( T' = \min\{t : S'_t = 0\} \) be the first time this path returns to zero. Define \( U'_t \) as
\[
U'_t = \begin{cases} 
-S'_t & \text{for } t \leq T \\
S'_t & \text{for } t > T.
\end{cases}
\]
This gives us the second required injection, proving the lemma.

Whilst the above result is straightforward, it provides the key step in overcoming the “objection” that arises in the following proof. For simplicity, we also introduce the following notation:
\[
N_{n,b} : \text{ the number of paths with } S_0 = 0 \text{ and } S_n = b.
\]

**Lemma 2.** Let \( \{S_k\}_{1 \leq k \leq n} \) be a simple random walk. Then the probability that the random walk does not return to zero is equal to the probability that the walk is 0 at time 2n, that is,
\[
P(S_1 \neq 0, \ldots, S_{2n} \neq 0) = P(S_{2n} = 0).
\]

**Proof.** Note that by the symmetry of random walks that are either all positive or all negative, we have
\[
P(S_1 \neq 0, \ldots, S_{2n} \neq 0) = P(S_1 > 0, \ldots, S_{2n} > 0) + P(S_1 < 0, \ldots, S_{2n} < 0),
\]
(1)
\[
= 2P(S_1 > 0, \ldots, S_{2n} > 0).
\]
A random walk with all positive values must satisfy \( S_{2n} = 2r \) for some \( r \) in \( \{1, \ldots, n\} \). This implies
\[
P(S_1 > 0, \ldots, S_{2n} > 0) = \sum_{r=1}^{n} P(S_1 > 0, \ldots, S_{2n-1} > 0, S_{2n} = 2r).
\]
(2)
We note that if \( S_1 > 0 \), then \( S_1 \) is necessarily 1. Hence the number of paths satisfying the right hand side condition in (2) is the total number of paths with \( S_1 = 1 \) and \( S_{2n} = 2r \) minus the number of paths with \( S_1 = 1 \) and \( S_{2n} = 2r \) which return to zero at least once. From the reflection principle, the latter term equals the total number of paths with \( S_1 = -1 \) and \( S_{2n} = 2r \). By translation, the number of paths with \( S_1 = 1 \) and \( S_{2n} = 2r \) equal the number of paths with \( S_0 = 0 \) and \( S_{2n-1} = 2r - 1 \). Thus we see the summands satisfy
\[
P(S_1 > 0, \ldots, S_{2n-1} > 0, S_{2n} = 2r) = [N_{2n-1,2r-1} - N_{2n-1,2r+1}]2^{-n},
\]
\[
= \frac{1}{2} [P(S_{2n-1} = 2r - 1) - P(S_{2n-1} = 2r + 1)].
\]
We observe that the right hand side of (2) is then a telescoping sum and the remaining negative probability \( P(S_{2n-1} = 2n + 1) \) is impossible. Using \( P(S_{2n-1} = 1) = P(S_{2n-1} = -1) \), we then compute
\[
P(S_1 > 0, \ldots, S_{2n} > 0) = \frac{1}{2} P(S_{2n-1} = 1)
\]
\[
= \frac{1}{2} \left[ \frac{1}{2} P(S_{2n-1} = 1) + \frac{1}{2} P(S_{2n-1} = -1) \right]
\]
\[
= \frac{1}{2} P(S_{2n} = 0).
\]
The result follows from combining this with equation (1).
With this result in hand, it is now straightforward to prove Feller’s arcsine law.

**Theorem 3** (Arcsine law). Let $L_{2n} = \sup\{2k \leq 2n : S_{2k} = 0\}$ be the last return to 0. Then the probability mass function of $L_{2n}$ is

$$P(L_{2n} = 2k) = P(S_{2k} = 0)P(S_{2n-2k} = 0).$$

Further, we have

$$P(L_{2n} = 2k) \rightarrow \frac{1}{n\pi \sqrt{x(1-x)}}$$

where $k/n \rightarrow x$ as $n \rightarrow \infty$.

**Proof.** For simplicity, let $S'_i = \sum_{j=1}^i X_{2k+j}$. We proceed by conditioning on the event $S_{2k} = 0$ and using independence between $S_{2k}$ and the $S'_i$.

$$P(L_{2n} = 2k) = P(S_{2k} = 0, S_{2k+1} \neq 0, \ldots, S_{2n} \neq 0),$$

$$= P(S_{2k} = 0, S'_1 \neq 0, \ldots, S'_{2n-2k} \neq 0),$$

$$= P(S_{2k} = 0)P(S'_1 \neq 0, \ldots, S'_{2n-2k} \neq 0).$$

Since $S'_i$ is a random walk, it has the same distribution as $S_i$. Thus by lemma 2 we conclude

$$P(L_{2n} = 2k) = P(S_{2k} = 0)P(S_{2n-2k} = 0).$$

For the second statement, we use Stirling’s approximation. For large $k$, we have the approximation

$$P(S_{2k} = 0) = \binom{2k}{k} 2^{-2k} \approx \frac{1}{\sqrt{\pi k}},$$

and the result follows. □

The name “arcsine law” follows from approximating the distribution of $L_{2n}$. Assuming a sufficiently large $n$, we have

$$P(L_{2n} \leq 2k) \approx \sum_{k<n} \frac{1}{n\pi \sqrt{x(1-x)}} ,$$

$$\approx \int_{0}^{x} \frac{1}{n\pi \sqrt{x(1-x)}} dx ,$$

$$= \frac{2}{\pi} \arcsin \sqrt{x} ,$$

where $x = k/n$.

One of the counter-intuitive results of such arcsine laws is that the distribution of $L_{2n}$ is symmetric about $n$, with a minimum at $P(L_{2n} = n)$ and much of the mass of the distribution in the tails.

**References**
