A SURVEY OF THE VITALI-HAHN-SAKS THEOREM WITH AN
APPLICATION IN PROBABILITY THEORY

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Abstract. Introduced by Vitali(1907), Hahn(1922) and Saks(1933), the Vitali-Hahn-Saks theorem plays a vital role in measure theory in proving some weak convergence theorems. In this Toolbox Essay, we will first introduce the Nikodym theorem followed by a self-contained proof. Next, we state the Vitali-Hahn-Saks theorem and show by an example how this theorem can be used in probability theory. The benefit of this Toolbox Essay is that it can help probabilists look into convergence problems from Vitali’s view. Our presentation here is based on James Brooks[1].

1. Introduction to Vitali-Hahn-Saks Theorem

The Vitali-Hahn-Saks (VHS) theorem is closely related to the Nikodym theorem which states the following:

**Theorem 1 (Nikodym).** Let \((X, \mathcal{A})\) be a measurable space and \(\{\mu_n\}\) be a sequence of bounded and possibly signed measures. Assume for any \(A \in \mathcal{A}\), we have

\[
\lim_{n \to \infty} \mu_n(A) = \mu(A)
\]

for some set function \(\mu\). Then \(\mu\) is a bounded measure and moreover \(\{\mu_n\}\) is equicontinuous in the sense that \(\mu_n(A_m) \to 0\) uniformly in \(n\) for any decreasing sequence of measurable sets for which \(\bigcap_{m=1}^{\infty} A_m = \emptyset\).

In the proof of the Nikodym theorem, we will need Schur’s lemma. This lemma is elementary but may have other applications.

**Lemma 2 (Schur).** Let \(\{c_{i,j}\}_{i,j=1,2,...} \subseteq \mathbb{C}\) have the following properties:

(a) \(\sum_{j=1}^{\infty} |c_{i,j}| < \infty\) for any \(i\),

(b) for any subset \(S \subseteq \mathbb{N}\), \(\lim_{i \to \infty} \sum_{j \in S} c_{i,j}\) exists.

Then there is a sequence \(\{c_j\} \subseteq \mathbb{C}\) such that \(\sum_{j=1}^{\infty} |c_j| < \infty\) and

\[
\lim_{i \to \infty} \sum_{j=1}^{\infty} |c_{i,j} - c_j| = 0.
\]

**Proof.** To begin with, we show that given any \(\{z_1, \cdots, z_n\} \subseteq \mathbb{C}\), there is \(S \subseteq \{j : 1 \leq j \leq n\}\) such that

\[
\sum_{j=1}^{n} |z_j| \leq 4\sqrt{2} \sum_{j \in S} |z_j|.
\]

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To see this, we divide the complex plane into its four diagonal quadrants $Q_1, Q_2, Q_3, Q_4$ in the way that $Q_1 = \{ z = x + yi : |y| \leq x, x \geq 0 \}$. Then, without loss of generality we may assume that

$$\sum_{z_j \in Q_1} |z_j| \geq \frac{1}{4} \sum_{j=1}^{n} |z_j|.$$ 

Notice that for $z_j \in Q_1$, we have $\sqrt{2} \text{Re}(z_j) \geq |z_j|$. Hence,

$$\left| \sum_{j \in Q_1} z_j \right| \geq \sum_{j \in Q_1} \text{Re}(z_j) \geq \frac{1}{\sqrt{2}} \sum_{j \in Q_1} |z_j| \geq \frac{1}{4\sqrt{2}} \sum_{j=1}^{n} |z_j|.$$ 

So $S = Q_1$ is the subset we are seeking for.

Next, we assume that all the limits in our condition (b) are 0. In this case, we claim that we can take $c_j = 0$ for all $j$. Suppose this is not the case. Then there is an $\varepsilon > 0$ and a subsequence $\{i_k : k = 1, 2, \ldots \}$ such that for all $k$, we have

$$\sum_{j=1}^{\infty} |c_{i_k,j}| > \varepsilon.$$

We will show that there are strictly increasing sequences $\{p_k \} \subseteq \mathbb{N}$ and $\{n_k \} \subseteq \{i_k \}$ such that for all $k$

(2) $\sum_{j=1}^{p_k} |c_{n_k,j}| < \frac{\varepsilon}{r}$

and

(3) $\sum_{j=p_{k+1}+1}^{\infty} |c_{n_k,j}| < \frac{\varepsilon}{r}$,

where $r > 2 + 8\sqrt{2}$. To see this, starting from any $p_1$, we have seen from (1) that for each $n$, there is $Y_n \subseteq \{0, \ldots, p_1 \}$ such that

$$\sum_{j=1}^{p_1} |c_{n,j}| \leq 4\sqrt{2} \sum_{j \in Y_n} |c_{n,j}|.$$ 

Notice that here we only have finitely many possibilities for $Y_n$. It follows from (b) that we obtain (2) for some $n_1$. Then we use our condition (b) to choose $p_2$ for which (3) is true. Continue this process. From (2) and (3) we see that

(4) $\sum_{j=p_{k+1}+1}^{p_{k+1}} |c_{n_k,j}| > \varepsilon - \sum_{j=1}^{p_k} |c_{n_k,j}| - \sum_{j=p_{k+1}+1}^{\infty} |c_{n_k,j}| > \varepsilon \left(1 - \frac{2}{r}\right).$

Now, using (1), for each $k$ we choose $S_k \subseteq \{j : p_k + 1 \leq j \leq p_{k+1}\}$ so that

$$4\sqrt{2} \sum_{j \in S_k} |c_{n_k,j}| \geq \sum_{j=p_k+1}^{p_{k+1}} |c_{n_k,j}|.$$
Let $S = \bigcup S_k$. We see that for each $k$

$$\left| \sum_{j \in S} c_{n_k,j} \right| \geq \left| \sum_{j \in S_k} c_{n_k,j} \right| - \sum_{j=1}^{p_k} |c_{n_k,j}| - \sum_{j=p_k+1}^{\infty} |c_{n_k,j}| \geq \varepsilon \left( \frac{1}{4\sqrt{2}} - \frac{1}{2r\sqrt{2}} - \frac{r}{2} \right) > 0.$$  

This is a contradiction to (b).

For the general case, we take an increasing sequence $\{m_i : i = 1, 2, \ldots\}$ and define $a_{i,j} = c_{m_{i+1},j} - c_{m_i,j}$.

Notice that $\{a_{i,j}\}$ satisfies our conditions and all the limits in (b) are 0. Hence, from what we have proved in the special case, we conclude that

$$\lim_{i \to \infty} \sum_{j=1}^{\infty} |c_{m_{i+1},j} - c_{m_i,j}| = 0.$$  

Since $\{m_i : i = 1, 2, \ldots\}$ is arbitrary, this implies that $\{c_i : c_i = (c_{i,1}, c_{i,2}, \ldots)\}$ is a Cauchy sequence in $l^1$. The proof is then followed from the completeness of $l^1$.  

Now, we can prove the Nikodym theorem.

**Proof.** It suffices to show $\mu(A_j) \to 0$ for any $\{A_j\} \subseteq A$ decreasing to $\emptyset$. Let $E_k = A_k \setminus A_{k+1}$. Thus $\{E_k\}$ is a disjoint family and $A_k = \bigcup_{j=k}^{\infty} E_j$. Define $c_{i,j} = \mu(E_j)$.

We now check that $\{c_{i,j}\}$ defined above satisfies the conditions in Schur’s lemma. First, we see that for each $i = 1, 2, \ldots$,

$$\sum_{j=1}^{\infty} |c_{i,j}| = \sum_{j=1}^{\infty} |\mu_i(E_j)| \leq \sum_{j=1}^{\infty} |\mu_i|(E_j) = |\mu_i|(A_1) < \infty,$$  

which shows that condition (a) is satisfied. For condition (b), we see that for $S \subseteq \mathbb{N}$,

$$\lim_{i \to \infty} \sum_{j \in S} c_{i,j} = \lim_{i \to \infty} \mu_i \left( \bigcup_{j \in S} E_j \right) = \mu \left( \bigcup_{j \in S} E_j \right).$$  

Therefore, it follows from Schur’s lemma that $c_j = \mu(E_j)$ for all $j$ and

$$\lim_{i \to \infty} \sum_{j=1}^{\infty} |\mu_i(E_j) - \mu(E_j)| = 0. \quad (5)$$  

Next, we will show that

$$\lim_{n \to \infty} \sum_{j=n}^{\infty} \mu(E_j) = 0. \quad (6)$$  

Notice that

$$\sum_{j=n}^{\infty} |\mu(E_j)| \leq \sum_{j=n}^{\infty} |\mu_i(E_j) - \mu(E_j)| + \sum_{j=n}^{\infty} |\mu_i(E_j)| \leq \sum_{j=1}^{\infty} |\mu_i(E_j) - \mu(E_j)| + \sum_{j=n}^{\infty} |\mu_i(E_j)|.$$
Thus, we have
\[
\lim_{n \to \infty} \sum_{j=n}^{\infty} |\mu(E_j)| \leq \sum_{j=1}^{\infty} |\mu_i(E_j) - \mu(E_j)| + \lim_{n \to \infty} |\mu_i(\bigcup_{j=n}^{\infty} E_j)|.
\]

Since \(\mu_i\) is bounded and \(\{A_n\}\) decreases to \(\emptyset\), we have
\[
\lim_{n \to \infty} \sum_{j=n}^{\infty} |\mu(E_j)| \leq \sum_{j=1}^{\infty} |\mu_i(E_j) - \mu(E_j)|.
\]

Therefore, (6) follows from (5).

Our next step is to show that
\[
\lim_{j \to \infty} (\mu_i(A_j) - \sum_{k=j}^{\infty} \mu(E_k)) = 0, \quad \text{uniformly in } i.
\]

Notice from (6) that this is certainly true for each \(i\) but may not uniformly. For any \(\varepsilon > 0\), by (5) we see that there is an \(I\) such that for all \(i > I\),
\[
\sum_{j=1}^{\infty} |\mu_i(E_j) - \mu(E_j)| < \varepsilon.
\]

Take \(J\) to be such that for all \(i = 1, \ldots, I\),
\[
\sum_{j=1}^{\infty} |\mu_i(E_j) - \mu(E_j)| < \varepsilon.
\]

Hence, for all \(j > J\) and \(i = 1, 2, \ldots\), we have
\[
|\mu_i(A_j) - \sum_{k=j}^{\infty} \mu(E_k)| = \left| \sum_{k=j}^{\infty} (\mu_i(E_k) - \mu(E_k)) \right| \leq \sum_{k=j}^{\infty} |\mu_i(E_k) - \mu(E_k)| < \varepsilon,
\]
which shows (7).

Finally, to finish the proof, we combine (7) and the fact that
\[
\lim_{i \to \infty} \left( \mu_i(A_j) - \sum_{k=j}^{\infty} \mu(E_k) \right) = \mu(A_j) - \sum_{k=j}^{\infty} \mu(E_k),
\]
then apply a double sequence argument to obtain
\[
\lim_{j \to \infty} \left( \mu(A_j) - \sum_{k=j}^{\infty} \mu(E_k) \right) = \lim_{i \to \infty} \lim_{j \to \infty} \left( \mu_i(A_j) - \sum_{k=j}^{\infty} \mu(E_k) \right).
\]

It follows from (7) that the right-hand side of (8) is 0. Hence, applying (6) again to the left-hand side of (8), we obtain our desired conclusion that \(\lim_{j \to \infty} \mu(A_j) = 0\).

The equicontinuity in the Nikodym theorem follows easily from (6) and (7). To end this proof, we show that \(\mu\) is \(\sigma\)-additive in the following: let \(\{B_n : n = 1, 2, \ldots\} \subseteq A\) be disjoint and \(A_j = \bigcup_{n=1}^{\infty} B_n\). Then, \(\{A_j\}\) decreases to \(\emptyset\) and
\[
\mu\left( \bigcup_{n=1}^{\infty} B_n \right) = \sum_{n=1}^{j} \mu(B_n) + \mu(A_{j+1}).
\]

Hence, the \(\sigma\)-additivity follows from the fact that \(\lim_{j \to \infty} \mu(A_j) = 0\). \(\square\)

Now, as a strengthen form of the Nikodym theorem, we state the Vitali-Hahn-Saks Theorem:
Theorem 3 (Vitali-Hahn-Saks). Let \((X, \mathcal{A})\) be a measurable space and \(\nu\) be a measure. Suppose that \(\{\mu_n\}\) is a sequence of bounded measure and

(a) for any \(A \in \mathcal{A}\), \(\lim_{n \to \infty} \mu_n(A) = \mu(A)\) exists,

(b) for all \(n = 1, 2, \ldots, \mu_n \ll \nu\), i.e. \(\mu_n\) is absolutely continuous with respect to \(\nu\).

Then \(\{\mu_n\}\) is uniformly absolutely continuous.

We will not prove this theorem in this essay. The proof of this theorem can be found in [1]. Notice that without any further effort, we can apply the Vitali-Hahn-Saks theorem to probability theory to obtain

Proposition 4. Given \((\Omega, \mathcal{F})\), let \(\{f_n\}\) be a sequence of probability measures. If for every \(A \in \mathcal{F}\), \(\lim_{n \to \infty} f_n(A) = f(A)\), then \(f\) is a probability measure.

In the second part of this Toolbox Essay, we will also need the following result:

Proposition 5. Given \((X, \Sigma, \mu)\), let \(\{f_n\} \subseteq L^1_\mu(X)\) have the property that for any \(E \in \Sigma\)

\[
\lim_{n \to \infty} \int_E f_n d\mu
\]

exists and is finite. Then for any \(\varepsilon > 0\), there is \(\delta > 0\) for which

\[
\mu(E) < \delta \Rightarrow \int_E |f_n| d\mu < \varepsilon.
\]

2. Application of the Vitali-Hahn-Saks theorem on weak convergence

One of the applications of the Vitali-Hahn-Saks theorem in convergence of random variables is that we obtain a necessary and sufficient condition between rough convergence and some refined convergence such as weak convergence. We first review the definition of a convergence type called "vague convergence".

Definition 6. A sequence \(\{\mu_n\}\) of probability measures converge vaguely to a probability measure \(\mu\) if and only if on \((\Omega, \mathcal{F})\),

\[
\lim_{n \to \infty} \int v d\mu_n = \int v d\mu \text{ for any } v \in C_0(X).
\]

Proposition 7. Suppose \(\mu_n\) and \(\mu\) are probability measures and they satisfy the above condition with restriction that \(v \in C_B(X)\), where \(C_B(X)\) is the space of continuous and bounded functions. Suppose \(\mu_n\) and \(\mu\) satisfy the following condition

\[
\lim_{n \to \infty} \mu_n(F) = \mu(F) \text{ for any } F \in \mathcal{F}.
\]

Then \(\mu_n\) converges to \(\mu\) in distribution.

Proof. By the Vitali-Hahn-Saks Theorem, we know that every \(\mu_n\) is absolutely continuous with respect to \(\lambda\) and so is \(\mu\). By the Portmanteau Theorem, we know that \(\mu_n\) converges to \(\mu\) in distribution. Hence, in this case, vague convergence implies convergence in distribution under a mild restriction. \(\square\)

References