Statistics 900: Probability, Optimization, Sequential Selection
An Eclectic Collection of Exercises

Each of these exercises provide a tool, trick, or a piece of philosophy that I think is useful for a creative mathematical scientist to have in mind.

PROBLEM 1. "Gordon's Theorem of the Alternative" states that we always has exactly one of the following:

- There is an x such that $Ax < 0$
- There is a $y \in \mathbb{R}^n_+$ such that $y^T A = 0$

COMMENT: There are many "Theorems of the Alternative" for linear systems and linear inequalities. A slew of these are due to Stiemke and some of these have extensions objects other than vector spaces. I may give an analytic proof of this in class.

PROBLEM 2. **Diagonal Convergence Theorem.** Suppose that X_n converges almost surely to X. Suppose that $\{X_n\}$ is dominated, i.e. there exists a Y such that $|X_n| \leq Y$ for all $n = 1, 2, ...$ Finally suppose that $\mathcal{F}_n$, $n = 1, 2, ...$ is a filtration and $\mathcal{F}_\infty = \sigma\{\cup \mathcal{F}_n\}$. Now consider the array $M_{i,j} = E(X_i \,|\, \mathcal{F}_j)$ and show that we have "diagonal convergence""

$$\lim_{n\to\infty} M_{n,n} = E(X \,|\, \mathcal{F}_\infty).$$

HINT AND COMMENT: It's probably a good idea to consider

$$Z_n = \sup_{m\geq n} |X_m - X|$$

and note that it goes to zero almost surely and in L^1. Also, you might want to think how you can extend this to a "near diagonal" convergence theorem.