



## An Elementary Proof of the Grothendieck Inequality

Ron C. Blei

*Proceedings of the American Mathematical Society*, Vol. 100, No. 1. (May, 1987), pp. 58-60.

Stable URL:

<http://links.jstor.org/sici?sici=0002-9939%28198705%29100%3A1%3C58%3AAEPOTG%3E2.0.CO%3B2-8>

*Proceedings of the American Mathematical Society* is currently published by American Mathematical Society.

---

Your use of the JSTOR archive indicates your acceptance of JSTOR's Terms and Conditions of Use, available at <http://www.jstor.org/about/terms.html>. JSTOR's Terms and Conditions of Use provides, in part, that unless you have obtained prior permission, you may not download an entire issue of a journal or multiple copies of articles, and you may use content in the JSTOR archive only for your personal, non-commercial use.

Please contact the publisher regarding any further use of this work. Publisher contact information may be obtained at <http://www.jstor.org/journals/ams.html>.

Each copy of any part of a JSTOR transmission must contain the same copyright notice that appears on the screen or printed page of such transmission.

---

JSTOR is an independent not-for-profit organization dedicated to creating and preserving a digital archive of scholarly journals. For more information regarding JSTOR, please contact [support@jstor.org](mailto:support@jstor.org).

# AN ELEMENTARY PROOF OF THE GROTHENDIECK INEQUALITY

RON C. BLEI

ABSTRACT. An elementary proof of the Grothendieck inequality is given.

Since its appearance in [2, pp. 59–64] and reformulation in [3, pp. 277–280], Grothendieck's fundamental inequality has enjoyed several restatements and proofs within various frameworks of analysis (detailed accounts of which appear in [4]). The purpose of this note is to give an elementary and self-contained proof of the inequality: the argument below, an adaptation of the proof given in [1], requires knowing only that the expectation of a product of independent random variables equals the product of their expectations.

Let  $\mathbf{R}^N$  denote the space of sequences of real numbers with finitely many nonzero terms.  $\mathbf{R}^N$  will be equipped with the usual inner product,

$$\langle x, y \rangle = \sum_n x(n)y(n), \quad x, y \in \mathbf{R}^N,$$

and Euclidean norm,

$$\|x\| = \langle x, x \rangle^{1/2}, \quad x \in \mathbf{R}^N.$$

$B$  will denote the unit ball in  $\mathbf{R}^N$ , i.e.  $B = \{x \in \mathbf{R}^N : \|x\| \leq 1\}$ .

**THEOREM (GROTHENDIECK'S INEQUALITY).** *Let  $(a_{mn})_{m,n=1}^\infty$  be an array of complex numbers which satisfies*

$$(1) \quad \left| \sum_{m,n=1}^N a_{mn} s_m t_n \right| \leq \max_{1 \leq m, n \leq N} |s_m| |t_n|$$

*for all sequences of complex numbers  $(s_m)_{m=1}^\infty$ ,  $(t_n)_{n=1}^\infty$ , and all integers  $N \geq 1$ . Then, for all sequences of vectors in  $\mathbf{R}^N$ ,  $(x_m)_{m=1}^\infty$ ,  $(y_n)_{n=1}^\infty$ ,*

$$(2) \quad \left| \sum_{m,n=1}^N a_{mn} \langle x_m, y_n \rangle \right| \leq K \max_{1 \leq m, n \leq N} \|x_m\| \|y_n\|$$

*for all  $N \geq 1$  and some universal constant  $K$ .*

To start, define a real-valued function on  $\mathbf{R}^N \times \mathbf{R}^N$  by

$$(3) \quad A(x, y) = \prod_n (1 + x(n)y(n)), \quad x, y \in \mathbf{R}^N,$$

and estimate

$$(4) \quad |A(x, y)| \leq e^{\sum \ln(1 + |x(n)y(n)|)} \leq e^{\sum |x(n)y(n)|} \leq e^{\|x\| \cdot \|y\|}.$$

Received by the editors January 2, 1986 and, in revised form, March 20, 1986.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 26D15, 46C99; Secondary 47A68.

The author was partially supported by NSF Grant MCS-8301659.

LEMMA 1. Suppose  $(a_{mn})_{m,n=1}^{\infty}$  satisfies the hypothesis of the theorem above. Then, for all sequences of vectors  $(x_m)_{m=1}^{\infty}, (y_n)_{n=1}^{\infty}$  in the unit ball of  $\mathbf{R}^N$ ,

$$\left| \sum_{m,n=1}^N a_{mn} A(x_m, y_n) \right| \leq e$$

for all  $N \geq 1$ .

PROOF. Let  $(Z_n)_{n=1}^{\infty}$  be a sequence of independent real-valued random variables on some probability space so that

$$(5) \quad \mathbf{E}(Z_n) = 0, \quad \mathbf{E}(Z_n^2) = 1, \quad \text{and} \quad |Z_n| = 1 \quad \text{a.s. for all } n.$$

( $\mathbf{E}$  denotes expectation;  $(Z_n)_{n=1}^{\infty}$  could be taken as the usual system of Rademacher functions.) Given  $x \in \mathbf{R}^N$ , define a random variable

$$F(x) = \prod_n (1 + ix(n)Z_n) \quad (i = \sqrt{-1}),$$

and estimate (by (4))

$$(6) \quad |F(x)| \leq \left( \prod_n (1 + x(n)^2) \right)^{1/2} \leq e^{\|x\|^2/2} \quad \text{almost surely.}$$

For any  $x, y \in \mathbf{R}^N$ ,

$$\begin{aligned} \mathbf{E}(F(x)\overline{F(y)}) &= \prod_n \mathbf{E}(1 + ix(n)Z_n)(1 - iy(n)Z_n) \quad (\text{by independence}) \\ &= \prod_n (1 + x(n)y(n)) \quad (\text{by (5)}) \\ &= A(x, y). \end{aligned}$$

Therefore, for any  $(x_m)_{m=1}^{\infty}, (y_n)_{n=1}^{\infty} \subset B$  and all  $N \geq 1$ ,

$$\begin{aligned} \left| \sum_{m,n=1}^N a_{mn} A(x_m, y_n) \right| &= \left| \sum_{m,n=1}^N a_{mn} \mathbf{E}(F(x_m)\overline{F(y_n)}) \right| \\ &\leq \mathbf{E} \left| \sum_{m,n=1}^N a_{mn} F(x_m)\overline{F(y_n)} \right| \\ &\leq e \quad (\text{by (1) and (6)}). \quad \text{Q.E.D.} \end{aligned}$$

Next, expand the product on the right-hand side of (3):

$$(7) \quad A(x, y) = 1 + \langle x, y \rangle + \cdots + \sum_{n_1 > \cdots > n_J} x(n_1) \cdots x(n_J) y(n_1) \cdots y(n_J) + \cdots$$

Let  $\{E_J\}_{J=2}^{\infty}$  be an infinite partition of the natural numbers  $\mathbf{N}$ , so that each  $E_J \subset \mathbf{N}$  is infinite. Let  $W_J$  be the  $J$ -dimensional wedge in  $\mathbf{N}^J$  given by

$$W_J = \{(n_1, \dots, n_J) \in \mathbf{N}^J : n_1 > \cdots > n_J\},$$

and set up a one-to-one correspondence between  $E_J$  and  $W_J$ ,  $J \geq 2$ :

$$n \in E_J \leftrightarrow (n_1, \dots, n_J) \in W_J.$$

Given an arbitrary  $x \in B$ , define a vector  $\phi(x) = (\phi(x)(n))_{n \in \mathbf{N}}$  in  $\mathbf{R}^{\mathbf{N}}$  by

$$\phi(x)(n) = x(n_1) \cdots x(n_J), \quad n \in E_J, \quad J = 2, \dots,$$

and estimate

$$\begin{aligned} (8) \quad \|\phi(x)\| &= \left( \sum_{J=2}^{\infty} \sum_{n \in E_J} (x(n_1) \cdots x(n_J))^2 \right)^{1/2} \\ &\leq \left( \sum_{J=2}^{\infty} \frac{1}{J!} \left( \sum_{n \in \mathbf{N}} x(n)^2 \right)^J \right)^{1/2} \leq (e-2)^{1/2} \equiv \delta < 1. \end{aligned}$$

Write

$$\phi_{\delta}(x) = \phi(x)/\delta, \quad x \in B,$$

and, by the estimate above, note that  $\phi_{\delta}$  is a map from  $B$  into  $B$ . Solving for  $\langle x, y \rangle$  in (7),  $x, y \in B$ , we obtain

$$(9) \quad \langle x, y \rangle = A(x, y) - 1 - \delta^2 \langle \phi_{\delta}(x), \phi_{\delta}(y) \rangle.$$

Therefore, applying (9) recursively, we obtain for each  $J > 0$

$$(10) \quad \langle x, y \rangle = \sum_{j=0}^J (-\delta^2)^j [A(\phi_{\delta}^j(x), \phi_{\delta}^j(y)) - 1] + (-\delta^2)^{J+1} \langle \phi_{\delta}^{J+1}(x), \phi_{\delta}^{J+1}(y) \rangle$$

( $\phi_{\delta}^j$  denotes the  $j$ th iterate of  $\phi_{\delta}$ ). Finally, letting  $J \rightarrow \infty$  in (10), we deduce

LEMMA 2. For all  $x, y \in B$

$$\langle x, y \rangle = \sum_{j=0}^{\infty} (-\delta^2)^j [A(\phi_{\delta}^j(x), \phi_{\delta}^j(y)) - 1].$$

PROOF OF GROTHENDIECK'S INEQUALITY. It suffices to establish (2) for  $(x_m)_{m=1}^{\infty}, (y_n)_{n=1}^{\infty} \subset B$ . By Lemmas 1 and 2, we estimate

$$\begin{aligned} \left| \sum_{m,n=1}^N a_{mn} \langle x_m, y_n \rangle \right| &\leq \sum_{j=0}^{\infty} \delta^{2j} \left| \sum_{m,n=1}^N a_{mn} [A(\phi_{\delta}^j(x_m), \phi_{\delta}^j(y_n)) - 1] \right| \\ &\leq \sum_{j=0}^{\infty} \delta^{2j} (e+1) = (e+1)/(3-e). \quad \text{Q.E.D.} \end{aligned}$$

## REFERENCES

1. R. C. Blei, *A uniformity property for  $\Lambda(2)$  sets and Grothendieck's inequality*, Sympos. Math. **22** (1977), 321–336.
2. A. Grothendieck, *Résumé de la théorie métrique des produits tensoriels topologique*, Bol. Soc. Mat. São Paulo **8** (1956), 1–79.
3. J. Lindenstrauss and A. Pelczynski, *Absolutely summing operators in  $\mathcal{L}^p$ -spaces and their applications*, Studia Math. **29** (1968), 275–326.
4. G. Pisier, *Factorization of linear operators and geometry of Banach spaces*, CBMS Regional Conf. Ser. in Math. vol. 60, Amer. Math. Soc., Providence, R.I., 1986.

DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF CONNECTICUT, STORRS, CONNECTICUT 06268