Statistics 955: Stochastic Calculus and Financial Applications

Homework No. 7

Instructions. Complete your reading of Chapters 8 and 9. Keep a “list” of SDEs you can solve and keep notes about associated techniques of solution.

Problem 1. Consider a sequence of independent Bernoulli random variables with $P(X_i = 1) = p$, $P(X_i = -1) = q$, and $p < q$. Let $S_n$ denote the partial sum. Find an exact formula for $P(\max_n S_n \geq m)$. Hint: For BM with negative drift we used the solution of the ruin problem to show that the maximal process had the exponential distribution. This problem asks you to discover the (very pretty!) discrete time analog. Don’t forget that we already know the solution of the discrete time ruin problem.

Problem 2. Use the method of coefficient matching (p. 137–139) to solve the SDE

$$dX_t = -\frac{1}{2}X_t \, dt + \sqrt{1 - X_t^2} \, dB_t \quad \text{where} \quad X_0 = 0.$$  

Since the integrand is not Lipschitz for $X_t$ near $\pm 1$, our existence and uniqueness theorems do not “officially” apply. Still, once you get the “formal” answer, you’ll see that if we wanted to we could prove that our formal answer is an honest answer. One typically omits this work, but sometimes it is necessary. You may need to review “separable” ODEs. This is a good idea even if you can “guess” the solution.

Problem 3. Exercise 8.5 of the text gives a famous example of a nice process that is a local martingale but not a martingale. Do parts (a), (b), (c) of that exercise and then add as part (d) the following questions: Is this process $M_t$ bounded by some constant $B$ with probability one? Is the process $M_t$ a super-martingale? Finally — a question about question — why are these last two questions “natural”?

Problem 4. Do Exercise 9.4 of the text. It shows how for many Markov diffusions $X_t$ one can find a monotone $f$ such that $Y_t = f(X_t)$ satisfies an SDE of the form

$$Y_t = Y_0 + \int_0^t b(Y_s) \, ds + B_t.$$  

By the way, does this representation give you a quick way to see the value of the quadratic variation $\langle Y, Y \rangle_t$.

Problem 5. A Brownian bridge can be characterized by the SDE,

$$dX_t = -\frac{X_t}{1-t} \, dt + dB_t, \quad \text{for} \ 0 < t < 1 \quad \text{and} \quad X_0 = 0.$$

- Find a PDE such that if $f(x)$ satisfies the PDE, then $M_t = f(X_t)$ is a local martingale.
- Show that your PDE has $f(x) \equiv C$ where $C$ is a constant as its only solution.
- Give a heuristic probabilistic explanation (a gambler’s explanation) why one should not expect to find any non-trivial $f(x)$ such that $M_t = f(X_t)$ is a martingale (or a local martingale).

Coaching. Get the best handle that you can on the “theory” in the text, but be certain that you have mastered the computational bits. Most people learn how to do interesting computations first. More nuanced issues of theory make more sense once one knows what can be done with the theory. Definitely do not make yourself paranoid about “local martingales” versus “honest martingales.” Your first steps should always be to “Calculate Boldly!” When you get something interesting, then you can ask: “Now that I have something good, can I justify my steps?”