so M_t converges with probability one. If M_∞ denotes the value of this limit, then by Fatou's lemma and the bound $E(|M_t|) \le B$ we have $E(|M_\infty|) \le B$.

Incidentally, a sequence such as $\{\tau_n\}$ is relate to the notion of a a *localizing sequence*. In this particular problem, $\{\tau_n\}$ helps us to *localize* a problem concerning the large set $L^1(dP)$ to one that deals with the smaller set $L^2(dP) \subset L^1(dP)$.

Chapter 5

Solution for Problem 5.1. If we choose α such that $\alpha/\lambda < \epsilon$ and if we let $\tau_\alpha = \min\{t : B_t \ge \alpha\}$, then we have $\{\tau_\alpha < \alpha/\lambda\} \subset \{\tau < \epsilon\}$. Taking complements, we have

$$\begin{aligned} P(\tau \ge \epsilon) &\le P(\tau_\alpha \ge \alpha/\lambda) \\ &= 2\Phi(\alpha/\sqrt{\alpha/\lambda}) - 1 \qquad \text{by (5.22), page 90.} \end{aligned}$$

Since $\Phi(\alpha/\sqrt{\alpha/\lambda}) \to 1/2$ as $\alpha \to 0$ we therefore find $P(\tau \ge \epsilon) = 0$.

Solution for Problem 5.2. By interchanging expectation and integration, it is immediate that $E(U_t) = E(V_t) = 0$. Next, since $X_t = \sqrt{t}Z$, direct integration gives $V_t = (2/3)t^{3/2}Z$, so $\mathrm{Var}(V_t) = (4/9)t^3$. Finally, we have

$$\begin{aligned} \mathrm{Var}(U_t) = E(U_t^2) = E\left[\int_0^t B_s\,ds \int_0^t B_u\,du\right] &= \int_0^t \int_0^t E(B_s B_u)\,ds\,du \\ = \int_0^t \int_0^t \min(s,u)\,ds\,du &= 2\int_0^t \int_0^u s\,ds\,du = t^3/3. \end{aligned}$$

Solution for Problem 5.3. First write the integral as a sum:

$$\begin{aligned} \int_0^1 B_t\,dt &= \sum_{k=0}^{n-1} \int_{k/n}^{(k+1)/n} B_t\,dt \\ &= \frac{1}{n}\sum_{k=0}^{n-1} B_{k/n} + \sum_{k=0}^{n-1} \int_{k/n}^{(k+1)/n} (B_t - B_{k/n})\,dt. \end{aligned} \tag{15.51}$$

The processes $\{B_{k/n} : k = 0, 1, ...n\}$ and $\{n^{-\frac{1}{2}} S_k : k = 0, 1, ...n\}$ have the same joint distributions, so the first sum in (15.51) is equal in distribution to A_n. Thus, it suffices to show that the second sum of (15.51) converges to zero in probability. We begin by noting

$$r_k(n) \stackrel{\text{def}}{=} \int_{k/n}^{(k+1)/n} (B_t - B_{k/n})\,dt \stackrel{\text{d}}{=} \int_0^{1/n} B_t\,dt,$$

and this gives us the bound

$$E(|r_k(n)|) \le \int_0^{1/n} E(|B_t|)\,dt = n^{-3/2}\frac{2}{3}\sqrt{\frac{2}{\pi}}.$$

Summing these estimates gives $E(|r_0(n)+r_1(n)+\cdots+r_{n-1}(n)|)=O(n^{-\frac{1}{2}})$, so by Markov's inequality the second sum of (15.51) does converge to zero in probability.

Solution for Problem 5.4. Since $\max_{0\le t\le T} B_t$ has the same distribution as $|B_T|$, the first identity comes from

$$E(|B_T|)=T^{\frac{1}{2}}E(|B_1|)=T^{\frac{1}{2}}\frac{2}{\sqrt{2\pi}}\int_0^\infty xe^{x^2/2}\,dx=\sqrt{\frac{2T}{\pi}}.$$

Since $\max_{0\le t\le T} B_t$ and $-\min_{0\le t\le T} B_t$ have the same distribution and since $\max_{0\le t\le T}|B_t|\le \max_{0\le t\le T} B_t-\min_{0\le t\le T} B_t$ the second assertion of (5.44) follows from the first.

Solution for Problem 5.5. If we condition on the value of B_s and use the reflection principle for the Brownian motion $X_u=B_u-B_s$, $u>s$, then the left side of equation (5.45) becomes

$$\int_{-\infty}^0 P(\max_{u\in[s,t]}(B_u-B_s)>-x)e^{-x^2/2s}\frac{dx}{\sqrt{2\pi s}}=\int_{-\infty}^0 2P(B_t-B_s>-x)e^{-x^2/2s}\frac{dx}{\sqrt{2\pi s}},$$

and, again by conditioning, this equals the right side of equation (5.45).

Solution for Problem 5.6. If we set $G=\{(u,v)\in D: v-u\le x\}$ and $w=v-u$, then the density (5.18) gives us

$$\begin{aligned}P(B_t^*-B_t\le x)&=\int_G f(u,v)dudv=\int_0^x\int_0^\infty \frac{2(v+w)}{\sqrt{2\pi t^3}}\exp(-(v+w)^2/2)dvdw\\&=2\int_0^x \exp(-w^2/2)dw/\sqrt{2\pi t}=2\left(\Phi(x/\sqrt{t})-\frac{1}{2}\right)=P(B_t^*\le x).\end{aligned}$$

Since $X_t=B_t^*-B_t$ can decrease but B_t^* cannot, we see that these processes are not equivalent.

Solution for Problem 5.7. To prove $E(\tau)=\sigma^2$, we first condition on I and J and then we exploit the hitting time formula. Specifically, we have

$$\begin{aligned}E[\tau]&=E[E[\tau\,|\,I,J]]=E[(-IJ)]\\&=-\int_{-\infty}^0\int_0^\infty ts\gamma^{-1}(t-s)dF(s)dF(t)\\&=-\int_{-\infty}^0\int_0^\infty t^2s\gamma^{-1}dF(s)dF(t)+\int_{-\infty}^0\int_0^\infty ts^2\gamma^{-1}dF(s)dF(t)\\&=\int_0^\infty t^2dF(t)+\int_{-\infty}^0 s^2dF(s)=\sigma^2.\end{aligned}$$

To prove $P(B_\tau\le x)=F(x)$, we again use conditioning to get

$$
\begin{aligned}
P(B_\tau \le x) =& E[P(B_\tau \le x) \,|\, I, J]] = E\left[\mathbb{I}(I \le x)\frac{J}{J-I} + \mathbb{I}(J \le x)\frac{(-I)}{J-I}\right] \\
&= \int_{-\infty}^{0}\int_{0}^{\infty} \mathbb{I}(s \le x)\frac{t}{t-s}\gamma^{-1}(t-s)dF(s)dF(t) \\
&\qquad + \int_{-\infty}^{0}\int_{0}^{\infty} \mathbb{I}(t \le x)\frac{(-s)}{t-s}\gamma^{-1}(t-s)dF(s)dF(t) \\
&= \int_{-\infty}^{0} \mathbb{I}(s \le x)dF(s) + \int_{0}^{\infty} \mathbb{I}(t \le x)dF(t) \\
&= \int_{-\infty}^{\infty} \mathbb{I}(u \le x)dF(u) = F(x).
\end{aligned}
$$

Solution for Problem 5.8. From the set inclusions

$$\{Y_n \le x - \epsilon\} \subset \{X_n \le x\} \cup \{|Y_n - X_n| \ge \epsilon\}$$

$$\{X_n \le x\} \subset \{Y_n \le x + \epsilon\} \cup \{|Y_n - X_n| \ge \epsilon\}$$

we have the probability bounds

$$F(x - \epsilon) \le P(X_n \le x) + P(|Y_n - X_n| \ge \epsilon)$$

$$P(X_n \le x) \le F(x + \epsilon) + P(|Y_n - X_n| \ge \epsilon),$$

so for all $\epsilon > 0$ we have

$$F(x - \epsilon) \le \liminf_{n\to\infty} P(X_n \le x) \le \limsup_{n\to\infty} P(X_n \le x) \le F(x + \epsilon).$$

Solution for Problem 5.9. We relate B_t to $M_t = \max_{s\in[0,t]} |B_s|$ by noting

$$
\begin{aligned}
P(M_t \ge x) &= P(M_t \ge x,\, |B_t| \le x) + P(M_t \ge x,\, |B_t| \ge x) \\
&= P(M_t \ge x,\, |B_t| \le x) + P(|B_t| \ge x),
\end{aligned}
$$

so we then have

$$
\begin{aligned}
P(M_t \le x) &= 1 - P(|B_t| \ge x) - P(M_t \ge x,\, |B_t| \le x) \\
&= P(|B_t| \le x) - P(M_t \ge x,\, |B_t| \le x).
\end{aligned}
$$

Next, we set $\tau = \min\{t : |B_t| \ge x\}$ and $\tau_{-x} = \min\{t : B_t = -x\}$, so we have $\tau = \min\{t : |B_t| = x\}$. We then let $\widetilde{B_t}$ denote the reflection of B_t at x or at $-x$ according to which ever is hit first; that is, if $\tau_x < \tau_{-x}$ then $\widetilde{B}_t$ is the reflection of B_t at x while if $\tau_{-x} < \tau_x$ then $\widetilde{B_t}$ is the reflection of B_t at $-x$. We then note that $P(M_t \ge x,\, |B_t| \le x)$ is equal to

$$
\begin{aligned}
&P(\tau < t,\, \tau_x < \tau_{-x},\, B_t \in [-x,x]) + P(\tau < t,\, \tau_{-x} < \tau_x\, B_t \in [-x,x]) \\
&\quad = P(\tau < t,\, \tau_x < \tau_{-x},\, \widetilde{B_t} \in [x,3x]) + P(\tau < t,\, \tau_{-x} < \tau_x,\, \widetilde{B_t} \in [-3x,-x]) \\
&\quad = P(\tau < t,\, \tau_x < \tau_{-x},\, B_t \in [x,3x]) + P(\tau < t,\, \tau_{-x} < \tau_x,\, B_t \in [-3x,-x]) \\
&\quad = P(B_t \in [x,3x]) - P(\tau_{-x} < \tau_x,\, B_t \in [x,3x]) \\
&\qquad\qquad + P(B_t \in [-3x,-x]) - P(\tau_x < \tau_{-x},\, B_t \in [-3x,-x]) \\
&\quad = P(|B_t| \in [x,3x]) - T,
\end{aligned}
$$

where T is just shorthand for the two term sum

$$
P(\tau_{-x} < \tau_x,\, B_t \in [x,3x]) + P(\tau_x < \tau_{-x},\, B_t \in [-3x,-x]).
$$

In the first line of this calculation, we used the definition of the reflected process $\widetilde{B_t}$, and in the second line we used the equivalence of the processes $\{\widetilde{B_t}\}$ and $\{B_t\}$. We now repeat the argument for T to find

$$
\begin{aligned}
T &= P(\tau_{-x} < \tau_x,\, \widetilde{B_t} \in [-5x,-3x]) + P(\tau_x < \tau_{-x},\, \widetilde{B_t} \in [3x,5x]) \\
&= P(\tau_{-x} < \tau_x,\, B_t \in [-5x,-3x]) + P(\tau_x < \tau_{-x},\, B_t \in [3x,5x]) \\
&= P(B_t \in [-5x,-3x]) - P(\tau_x < \tau_{-x},\, B_t \in [-5x,-3x]) \\
&\qquad + P(B_t \in [3x,5x]) - P(\tau_{-x} < \tau_x,\, B_t \in [3x,5x]) \\
&= P(|B_t| \in [3x,5x]) - U,
\end{aligned}
$$

where $U = P(\tau_x < \tau_{-x},\, B_t \in [-5x,-3x]) + P(\tau_{-x} < \tau_x,\, B_t \in [3x,5x])$.

If we repeat our argument for U, then we find

$$
U = P(|B_t| \in [5x,7x]) - V,
$$

where $V = P(\tau_{-x} < \tau_x,\, B_t \in [5x,7x]) + P(\tau_x < \tau_{-x},\, B_t \in [-7x,-5x])$, and by continuing in this way we generate all the terms of the series (5.47).

Solution for Problem 5.10. From the general wavelet expansion (5.3)–(5.4), page 81, we have for Brownian motion that

$$
B_s - B_t = \sum_{j=0}^{\infty} D_j(s,t) \quad \text{where} \quad D_j(s,t) = \sum_{2^j \le n < 2^{j+1}} c_n \{\Delta_n(s) - \Delta_n(t)\},
$$

and we also have general bounds for the summands,

$$
|D_j(s,t)| \le \begin{cases} \max_{2^j \le n < 2^{j+1}} c_n \\ 2^{j+1}|s-t| \max_{2^j \le n < 2^{j+1}} c_n. \end{cases}
$$

The formula (3.10) for c_n for Brownian motion and Lemma 3.2, (both on page 48), then give us a random variable C such that $P(C < \infty) = 1$ for which

$$
|D_j(s,t)| \le \begin{cases} C(\omega)(j+1)^{\frac{1}{2}} 2^{-j/2} \\ C(\omega)|s-t|(j+1)^{\frac{1}{2}} 2^{j/2}. \end{cases}
$$

Breaking the sum into two parts and using our second estimate for $|D_j(s,t)|$ on the first part, we find for all m that

$$|B_{t+h} - B_t| \le C(\omega) \sum_{0\le j\le m} h(j+1)^{\frac{1}{2}} 2^{j/2} + C(\omega) \sum_{j:j>m} (j+1)^{\frac{1}{2}} 2^{-j/2}.$$

Comparison with geometric series give us constants C_1 and C_2 such that

$$\sum_{0\le j\le m} (j+1)^{\frac{1}{2}} 2^{j/2} \le 2^{m/2}(m+1)^{\frac{1}{2}} \sum_{0\le j\le m} 2^{(j-m)/2} \le C_1 2^{m/2}(m+1)^{\frac{1}{2}}$$

$$\text{and} \quad \sum_{j:j>m} (j+1)^{\frac{1}{2}} 2^{-j/2} \le C_2(m+1)^{\frac{1}{2}} 2^{-m/2},$$

so for $C'(\omega) = C(\omega)\max(C_1, C_2)$ we have for all $m = 1, 2, ...$ that

$$|B_{t+h} - B_t| \le C'(\omega)(m+1)^{\frac{1}{2}}(h2^{m/2} + 2^{-m/2}).$$

To complete the proof we then take $m = \lceil \log_2(1/h) \rceil$.

Chapter 6

Solution for Problem 6.1. For part (a) it suffices to note that

$$\{\omega : g(\omega) \le x\} = \bigcap_k \bigcup_N \bigcap_{n:\, n\ge N} \{\omega : g_n(\omega) \le x + 1/k\},$$

while for part (b), it suffices to take $g_n(\omega)$ to be the smallest value of $k/2^n$ such that $g(\omega) \le k/2^n$. For part (c), first choose f_n and g_n as in part (b) and note that $g_n f_n$ is $\mathcal{G}$ measurable since

$$\{\omega : f_n(\omega)g_n(\omega) \le x\} = \bigcup \{\omega : f_n(\omega) = k/2^n\} \cap \{\omega : g_n(\omega) = j/2^n\}$$

where the union is over all j and k such that $(j/2^n)(k/2^n) \le x$. Since fg is the limit of $f_n g_n$ we see that fg is measurable by part (a).

Solution for Problem 6.2. In general, to show that $f(\omega,t) = \mathbb{I}(t \le \tau(\omega))$ is progressively measurable, it suffices to find f_n progressively measurable such that $f_n(\omega,t)$ converges to f for all $\omega \in \Omega$ and all $0 \le t \le T$. Here we do this by approximating τ. If we fix $0 \le t \le T$ and let $\tau_n(\omega)$ be the smallest value of $kt/2^n$ such that $\tau(\omega) \le kt/2^n$, then τ_n is again an stopping time. We also have the representation

$$S \equiv \{(\omega, s) \in \Omega \times [0,t] : \tau_n(\omega) < s\} = \bigcup_{0\le k\le 2^n} \{\omega : \tau_n = kt/2^n\} \times (kt/2^n, t].$$

For each $0 \le k \le 2^n$ we have $\{\omega : \tau_n = kt/2^n\} \in \mathcal{F}_t$ since τ_n is a stopping time. Thus, S is a countable union of rectangles $A \times B$ with A in $\mathcal{F}_t$ and with B a subinterval of $[0,t]$. Thus S is an element of the product σ-field $\mathcal{F}_t \times \mathcal{B}([0,t])$, and this is what we need to show $(\omega, s) \mapsto \mathbb{I}(\tau_n(\omega) < s)$ is $\mathcal{I}_t$-measurable. Finally, for all (ω, s) we have as $n \to \infty$ that $\mathbb{I}(\tau_n(\omega) < s)$ converges to $\mathbb{I}(\tau(\omega) < s)$. Thus, by Exercise 6.1 part (a) we see that $(\omega, s) \mapsto \mathbb{I}(\tau(\omega) < s)$ is $\mathcal{I}_t$-measurable. Moreover, by Exercise 6.1 part (b), $(\omega, s) \mapsto f(s, \omega)\mathbb{I}(\tau(\omega) < s)$ is $\mathcal{I}_t$-measurable. This holds for each $0 \le t \le T$, so $g(\omega, t) = f(\omega, t)\mathbb{I}(t < \tau)$ is progressively measurable.

Solution for Problem 6.3. First fix $t \in [0, T]$ and take $f_n(\omega, s)$ equal to $f(\omega, (k+1)t/2^n)$ whenever $s \in (kt/2^n, (k+1)t/2^n]$ and $0 \le k < 2^n$. To check that the map $f_n : \Omega \times [0, t] \to \mathbb{R}$ is $\mathcal{F}_t \times \mathcal{B}([0, t])$-measurable, we note that

$$\{(\omega, s) : f_n(\omega, s) \le x\} = \cup_k \{\omega : f(\omega, (k+1)t/2^n) \le x\} \times (kt/2^n, (k+1)t/2^n]$$

is a countable union of $\mathcal{F}_t \times \mathcal{B}([0, t])$ rectangles. By the right-continuity of f we have $f_n(\omega, s) \to f(\omega, s)$ for all (ω, s), and the limit of a sequence of $\mathcal{F}_t \times \mathcal{B}([0, t])$-measurable functions is $\mathcal{F}_t \times \mathcal{B}([0, t])$-measurable. Thus, for each $t \in [0, T]$ the map $f : \Omega \times [0, t] \to \mathbb{R}$ is $\mathcal{F}_t \times \mathcal{B}([0, t])$-measurable, so f is progressively measurable.

Solution for Problem 6.5. Taking the hint, we expand the isometry relation $\|I(f+g)\|^2_{L^2(dP)} = \|f+g\|^2_{L^2(dP \times dt)}$ then remove the squared terms by subtracting $\|I(f)\|^2_{L^2(dP)} = \|f\|^2_{L^2(dP \times dt)}$ and $\|I(g)\|^2_{L^2(dP)} = \|g\|^2_{L^2(dP \times dt)}$.

Solution for Problem 6.6. Both X_t and Y_t have mean zero, so Itô's isometry gives $\mathrm{Var}(X_t) = (2/3)(2/\pi)^{1/2} t^{3/2}$ and $\mathrm{Var}(Y_t) = t^3 + 3t^4/2 + t^5/5$. These computations are simplified by recalling that B_s has the same distribution as $\sqrt{s}Z$ where $Z \sim N(0, 1)$ and by using the known moments of Z.

Solution for Problem 6.7. The proof just requires two observations. First, by the argument given in equations (6.20)-(6.22) one finds that

$$\sum_{k=1}^{n} B_{(k-1)/n}(B_{k/n} - B_{(k-1)/n}) \text{ converges in } L^2(dP) \text{ to } \int_0^1 B_t \, dB_t.$$

Second, we have $\epsilon_k/\sqrt{n} \overset{\mathrm{d}}{=} B_{k/n} - B_{(k-1)/n}$ and $S_{k-1}/\sqrt{n} \overset{\mathrm{d}}{=} B_{(k-1)/n}$ together with the corresponding statements for all of the joint distributions.

Solution for Problem 6.8. For $f(\omega, s) = |B_s|^{\frac{1}{2}}$ the martingale (6.27), page 114, and Doob's stopping time theorem, tell us $\mathrm{Var}(X) = 1$. Alternatively, one can apply the Itô isometry to $f(\omega, s) = |B_s|^{\frac{1}{2}} \mathbb{I}(s \le \tau)$

Solution for Problem 6.9. Given $g \in \mathcal{C}_B$, we take $t_i = iT/n$ for $0 \le i \le n$ and set

$$g_n(\omega, t) = \sum_{i=0}^{n-1} g(\omega, t_i)\mathbb{I}(t_i < t \le t_{i+1}). \tag{15.52}$$

Since $g(\omega, t_i) \in \mathcal{F}_{t_i}$ we see that $g_n \in \mathcal{H}_0^2$ by the definition of $\mathcal{H}_0^2$ (page 104). We also have for each fixed ω that

$$\sup_{0\le t\le T} |g_n(\omega,t) - g(\omega,t)| \le \sup_{s,t:|s-t|\le 1/n} |g(\omega,s) - g(\omega,t)| \equiv \mu_n(\omega).$$

By the uniform continuity of $t \mapsto g(\omega,t)$ on $[0,T]$, one has for each ω that $\mu_n(\omega) \to 0$ as $n \to \infty$. Two applications of the DCT then show that $g_n \to g$ in $L^2(dP \times dt)$.

Chapter 7

Solution for Problem 7.1. We argue by contradiction. If the limit were not infinite, then since $\tau_M(\omega)$ is nondecreasing we would have

$$\lim_{M\to\infty} \tau_M(\omega) = t^* < \infty.$$

The continuity of f then implies $f(t^*) = \infty$, but this contradicts the continuity of f. The corollary is then immediate since $f(B_t)$ is continuous on a set of probability one.

Look out for known bug re: such stopping times.

Solution for Problem 7.2. Consider the nondecreasing stopping times defined by setting $\nu_n(\omega) = \inf\{t : |X_t| \ge n, \text{ or } t \ge T\}$. For each ω, the mapping $t \mapsto X_t(\omega)$ is bounded on $[0,T]$ so for each ω there is an $N(\omega)$ such that $\nu_n(\omega) = T$ for all $n \ge N(\omega)$. Also, by Doob's stopping time theorem, $X_{t\wedge\nu_n}$ is a martingale, so by Jensen's inequality $\phi(X_{t\wedge\nu_n})$ is a submartingale. This says that $Y_{t\wedge\nu_n}$ is a submartingale. Thus, $\{\nu_n\}$ has both of the qualities needed to show that Y_t is a local submartingale. Finally, taking $X_t = B_t$ and $\phi(x) = e^{x^2}$ shows that Y_t need not be an honest submartingale; in this example $Y_t = \phi(B_t)$ is not even integrable.

Solution for Problem 7.3. By following the argument of Proposition 7.10 (page 136) through equation (7.24) we find in the present case that

$$X_{s\wedge\tau_k} \le E(X_{t\wedge\tau_k} \,|\, \mathcal{F}_s) \quad \text{for all } 0 \le s \le t \le T. \tag{15.53}$$

Since $\tau_k \to T$ as $k \to \infty$, continuity gives us $X_{s\wedge\tau_k} \to X_s$ and $X_{t\wedge\tau_k} \to X_t$. Finally, $|X_{t\wedge\tau_k}| \le X^* \in L^1(dP)$ so the bound (15.53) and the DCT give us the submartingale condition for $\{X_t\}$.

Solution for Problem 7.4. There are stopping times ν_n such that $\nu_n \to \infty$ with probability one and such that $t \mapsto M_{t\wedge\nu_n}$ is a martingale for each n. We then have $E(M_{t\wedge\nu_n}) = 0$, and $M_{t\wedge\nu_n} \ge 0$ so $P(M_{t\wedge\nu_n} = 0) = 1$. Now let $n \to \infty$.

Solution for Problem 7.5. Since M_t is a local martingale with $M_0 = 0$ there is a sequence of nondecreasing sequence of stopping times $\{\tau_n\}$ such that $\tau_n \to \infty$ with probability one and such that $t \mapsto M_{t\wedge\tau_n}$ is a martingale for each n. By Doob's stopping time theorem $M_{t\wedge\tau_n\wedge\tau}$ is also a martingale for each n, so we have $E(M_{t\wedge\tau_n\wedge\tau}) = 0$. By continuity $M_{t\wedge\tau_n\wedge\tau} \to 0$ as $n \to \infty$ so by the hypothesis on X and the DCT we have $E(M_{t\wedge\tau}) = 0$ as required.