$$\sup_{0\le t\le T} |g_n(\omega,t)-g(\omega,t)| \le \sup_{s,t:|s-t|\le 1/n} |g(\omega,s)-g(\omega,t)| \equiv \mu_n(\omega).$$

By the uniform continuity of $t \mapsto g(\omega,t)$ on $[0,T]$, one has for each ω that $\mu_n(\omega)\to 0$ as $n\to\infty$. Two applications of the DCT then show that $g_n \to g$ in $L^2(dP\times dt)$.

Chapter 7

Solution for Problem 7.1. We argue by contradiction. If the limit were not infinite, then since $\tau_M(\omega)$ is nondecreasing we would have

$$\lim_{M\to\infty} \tau_M(\omega) = t^* < \infty.$$

The continuity of f then implies $f(t^*)=\infty$, but this contradicts the continuity of f. The corollary is then immediate since $f(B_t)$ is continuous on a set of probability one.

Look out for known bug re: such stopping times.

Solution for Problem 7.2. Consider the nondecreasing stopping times defined by setting $\nu_n(\omega) = \inf\{t : |X_t| \ge n, \text{ or } t \ge T\}$. For each ω, the mapping $t\mapsto X_t(\omega)$ is bounded on $[0,T]$ so for each ω there is an $N(\omega)$ such that $\nu_n(\omega)=T$ for all $n\ge N(\omega)$. Also, by Doob's stopping time theorem, $X_{t\wedge\nu_n}$ is a martingale, so by Jensen's inequality $\phi(X_{t\wedge\nu_n})$ is a submartingale. This says that $Y_{t\wedge\nu_n}$ is a submartingale. Thus, $\{\nu_n\}$ has both of the qualities needed to show that Y_t is a local submartingale. Finally, taking $X_t = B_t$ and $\phi(x)=e^{x^2}$ shows that Y_t need not be an honest submartingale; in this example $Y_t=\phi(B_t)$ is not even integrable.

Solution for Problem 7.3. By following the argument of Proposition 7.10 (page 136) through equation (7.24) we find in the present case that

$$X_{s\wedge\tau_k} \le E(X_{t\wedge\tau_k}\,|\,\mathcal{F}_s) \quad \text{for all } 0\le s\le t\le T. \tag{15.53}$$

Since $\tau_k\to T$ as $k\to\infty$, continuity gives us $X_{s\wedge\tau_k}\to X_s$ and $X_{t\wedge\tau_k}\to X_t$. Finally, $|X_{t\wedge\tau_k}|\le X^*\in L^1(dP)$ so the bound (15.53) and the DCT give us the submartingale condition for $\{X_t\}$.

Solution for Problem 7.4. There are stopping times ν_n such that $\nu_n\to\infty$ with probability one and such that $t\mapsto M_{t\wedge\nu_n}$ is a martingale for each n. We then have $E(M_{t\wedge\nu_n})=0$, and $M_{t\wedge\nu_n}\ge 0$ so $P(M_{t\wedge\nu_n}=0)=1$. Now let $n\to\infty$.

Solution for Problem 7.5. Since M_t is a local martingale with $M_0=0$ there is a sequence of nondecreasing sequence of stopping times $\{\tau_n\}$ such that $\tau_n\to\infty$ with probability one and such that $t\mapsto M_{t\wedge\tau_n}$ is a martingale for each n. By Doob's stopping time theorem $M_{t\wedge\tau_n\wedge\tau}$ is also a martingale for each n, so we have $E(M_{t\wedge\tau_n\wedge\tau})=0$. By continuity $M_{t\wedge\tau_n\wedge\tau}\to 0$ as $n\to\infty$ so by the hypothesis on X and the DCT we have $E(M_{t\wedge\tau})=0$ as required.

Solution for Problem 7.6. For these processes to be equivalent it is necessary that $\mathrm{Var}(X_t) = \mathrm{Var}(Y_t)$, but we have

$$\mathrm{Var}(X_t) = \int_0^t e^{2s}\, ds = \frac{1}{2}e^{2t} \quad \text{and} \quad \mathrm{Var}(Y_t) = \tau_t,$$

so we *define* τ_t by setting $\tau_t = e^{2t}/2$. Since X_t and Y_t both have independent increments, this definition implies that

$$\mathrm{Cov}(X_s, X_t) = \min(e^{2s}/2, e^{2t}/2) \quad \text{and} \quad \mathrm{Cov}(Y_s, Y_t) = \min(e^{2s}/2, e^{2t}/2).$$

Since X_t and Y_t are both Gaussian process, the equality of the covariances for all $0 \le s \le t < \infty$ implies that the processes are equivalent. Finally, since Y_t is Gaussian with mean zero and variance $e^{2t}/2$ we have

$$\begin{aligned} &E(X_t^4) = E(Y_t^4) = 3(e^{2t}/2)^2 = (3/4)e^{4t} \quad \text{and} \\ &P(X_t \le 1) = P(Y_t \le 1) = \Phi(2^{\frac{1}{2}}e^{-t}). \end{aligned}$$

Solution for Problem 7.7. It is immediate that $\mathcal{F}_\tau$ contains the empty set and that $\mathcal{F}_\tau$ is closed under countable unions, so it remains only to consider complements. For $A \in \mathcal{F}_\tau$ we have $A \cap \{\tau \le t\} \in \mathcal{F}_t$ for each t, so we also have $A^c \cup \{\tau > t\} \in \mathcal{F}_t$ since $\mathcal{F}_t$ is closed under complements. We also have $\{\tau \le t\} \in \mathcal{F}_t$ since τ is a stopping time, and $\mathcal{F}_t$ is closed under intersection so

$$\begin{aligned} \big(A^c \cup \{\tau > t\}\big) \cap \{\tau \le t\} &= \big(A^c \cap \{\tau \le t\}\big) \cup \big(\{\tau > t\} \cap \{\tau \le t\}\big) \\ &= \big(A^c \cap \{\tau \le t\}\big) \cup \emptyset = A^c \cap \{\tau \le t\} \end{aligned}$$

is in $\mathcal{F}_t$ for each t. By Definition 7.3, this confirms that $A^c \in \mathcal{F}_\tau$.

Chapter 8

Solution for Problem 8.1. The pattern used to prove Theorem 8.1 needs no real changes, but a truly committed student may also want to justify the remainder bound (8.12). Since we only assume $f \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R})$ rather than $f \in C^{2,2}(\mathbb{R}^+ \times \mathbb{R})$, the traditional textbook versions of the multivariate Taylor expansion do not directly apply.

Solution for Problem 8.2. We apply Itô's formula to $f(t, x) = xh(t)$ and note $f_x = h(t)$, $f_{xx} = 0$, and $f_t = xh'(t)$.

Solution for Problem 8.3. By Itô's formula applied to $f(t, x) = tx$ we get

$$tB_t = \int_0^t s\, dB_s + \int_0^t B_s\, ds.$$

The first integral is a martingale, so we have $E(\tau B_\tau) = E(I)$ and consequently

$$E(I) = \frac{1}{3}AB(A - B) \quad \text{by equation (8.16).}$$

Solution for Problem 8.4. By Itô's formula we have

$$X_t^4 = 4\int_0^t X_s^3 \sigma(\omega,s)\,dB_s + 6\int_0^t X_s^2\sigma^2(\omega,s)\,ds. \tag{15.54}$$

The Itô isometry gives us $E(X_s^2) \le s$, but it is not a priori clear that the integrand $X_s^3\sigma(\omega,s)$ is in $\mathcal{H}^2$ so we cannot immediately say that the expectation of the first integral is zero. Thus, we introduce the localizer $\nu_N = \min\{u : |X_u| \ge N, \text{ or } u \ge T\}$. This makes the first integral a martingale, and we get the bound

$$E(X_{t\wedge\nu_N}^4) \le 6E\left(\int_0^{t\wedge\nu_N} X_s^2\sigma^2(\omega,s)\,ds\right) \le 6\int_0^t s\,ds = 3t^2.$$

The inequality (8.58) now follows by letting $N\to\infty$ and applying Fatou's lemma. An analogous argument gives one $E(X_t^{2k}) \le 1\cdot 3\cdot 5\cdots(2k-1)t^k$, and, with this in hand, Itô's formula and induction can be used to show that all of the odd moments of X_t vanish. Thus, as one might have guessed, all moments of X_t are dominated by the corresponding moments of B_t.

Solution for Problem 8.5. By Itô's formula applied to $f(t,\mathbf{x}) = t - |\mathbf{x}|^2/d$ we find that $M_t = t - |\mathbf{B}_t|^2/d$ is a local martingale. After checking that M_t is an honest martingale, we find by the familiar DCT and Doob stopping time arguments that $E(M_0) = E(M_\tau) = E(\tau) - d^{-1}E(|\mathbf{B}_\tau|^2)$. Since $|B_\tau| = R$ and $M_0 = -|\mathbf{x}_0|/d$, we then get our target formula (8.59).

Solution for Problem 8.6. For part (a) differentiate the first of the Cauchy–Riemann equations with respect to x and the second with respect to y. By equality of the crossed partial derivatives we see the sum of the resulting equations is zero. This proves that u is harmonic. Symmetry implies the same is true for v.

Taking the hint for (b), we note that $M_t = X_t^2 - Y_t^2$ is a martingale. The DCT and Doob stopping time arguments give us $E(M_0) = E(M_\tau)$ where τ is the first time that $\mathbf{B}_t$ hits either $H(1)$ or $H(5)$. We have $M_0 = 4$ and $E(M_\tau) = p\cdot 1 + (1-p)\cdot 5$ so $p = 1/4$.

Solution for Problem 8.7. For $0 \le s \le t$ we have martingale identity $E(M_t|\mathcal{F}_s) = M_s$. If we replace M_t and M_s by their series representations (8.61) and if we then interchange expectations and summations, then we find

$$\sum_{k=0}^\infty \alpha^k E\big(H_k(t,B_t)|\mathcal{F}_s\big) = \sum_{k=0}^\infty \alpha^k E\big(H_k(s,B_s)\big). \tag{15.55}$$

For the equality (15.55) to hold for all α in a neighborhood of zero, we have to have equality of the coefficients, and this will give us the martingale identity for each of the processes $t \mapsto H_k(t,B_t)$. For a truly complete proof, one should also justify the interchange operation that leads one to the identity (15.55).

Alternative, one can argue directly that the series representation (8.61) implies that $\partial H_k/\partial t + (1/2)\partial^2 H_k/\partial^2 x = 0$, $X_t = H_k(t, B_t)$ is a local martingale. Doob's maximal inequality will then show that $\sup_{0\le t\le T}|X_t|$ is integrable, so X_t is seen to be an honest martingale.

Solution for Problem 8.8. If we take $M_t = H_4(t, B_t) = B_t^4 - 6tB_t^2 + 3t^2$ and apply the usual DCT and Doob stopping time arguments, we find

$$0 = E(M_\tau) = E(B_\tau^4) - 6E(\tau B_\tau^2) + 3E(\tau^2).$$

Since we know $E(\tau) = A^2$ and since we have $|B_\tau| \equiv A$, we therefore find that $E(\tau^2) = (5/3)A^4$ and $\mathrm{Var}(\tau) = (2/3)A^4$. Incidentally, this calculation may seem a bit easier than the the method used in Exercise 4.9, page 76, but, at heart, the methods are the same.

Solution for Problem 8.9. Part (a) is routine, but it is still nice to note that once you find $f_{xx} = (-x^2 + y^2 + z^2)/(x^2 + y^2 + z^2)^{5/2}$, then you get $\Delta f = 0$ just by symmetry. For part (b), we note that the identity (8.62) and Jensen's inequality imply $E(M_t) \le 1/\sqrt{t}$. Since $E(M_1) = C > 0$ and since the mean is constant for a martingale, we see that M_t cannot be one.

Solution for Problem 8.10. By the PDE condition for $f(t, B_t)$ to be a local martingale (page 149) we find for $f(t, x) = e^{-\lambda t}\phi(x)$ that we need

$$0 = \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2} = -\lambda e^{-\lambda t}\phi(x) + \frac{1}{2}e^{-\lambda t}\phi''(x),$$

or $\phi''(x) - 2\lambda\phi(x) = 0$. By solving this equation, we see that for any $\phi(x)$ of the form $c_0\exp(x\sqrt{2\lambda}) + c_1\exp(-x\sqrt{2\lambda})$ the process $f(t, B_t)$ is a local martingale. One then directly checks that these are in fact honest martingales.

From the martingales $e^{-\lambda t}\exp(-B_t\sqrt{2\lambda})$ and $e^{\lambda t}\exp(-B_t\sqrt{2\lambda})$ and the familiar Doob stopping time argument (say as used to prove (4.13), page 66) we then have the twin identities

$$1 = E\big(e^{-\lambda\tau}\exp(-B_\tau\sqrt{2\lambda})\big) \quad \text{and} \quad 1 = E\big(e^{\lambda\tau}\exp(-B_\tau\sqrt{2\lambda})\big).$$

Therefore, if we introduce the related expectations $x = E\big(e^{-\lambda\tau}\mathbb{I}(B_\tau = A)\big)$ and $y = E\big(e^{-\lambda\tau}\mathbb{I}(B_\tau = -B)\big)$, we get the two relations

$$1 = e^{A\sqrt{2\lambda}}x + e^{-B\sqrt{2\lambda}}y \quad \text{and} \quad 1 = e^{-A\sqrt{2\lambda}}x + e^{B\sqrt{2\lambda}}y$$

which we may solve to find

$$x = \frac{\sinh(B\sqrt{2\lambda})}{\sinh((A+B)\sqrt{2\lambda})} \quad \text{and} \quad y = \frac{\sinh(A\sqrt{2\lambda})}{\sinh((A+B)\sqrt{2\lambda})}.$$

Finally we just sum these terms to get the elegant formula

$$E(e^{-\lambda\tau}) = x + y = \frac{\sinh(A\sqrt{2\lambda}) + \sinh(B\sqrt{2\lambda})}{\sinh((A+B)\sqrt{2\lambda})}.$$

This morsel deserves to be savored. For example, one should note that when $B \to \infty$ it converges to $\exp(-A\sqrt{2\lambda})$, and thus it recaptures the formula (4.12), page 65, for the Laplace transform of the hitting time of a line. Also, with a little arithmetic, one can also recapture the (rather different looking!) formula for the symmetric problem $A = B$ of Exercise 4.9, page 76. Finally, when this martingale derivation is compared with the much older discrete derivation of Skorohod (1962, pp. 163–166) one gets a clear sense of the power that the Itô calculus can provide.

Chapter 9

Solution for Problem 9.1. The drift of the SDE (9.37) is linear in X_t and has a deterministic dB_t coefficient, so it is a candidate for coefficient matching with a product process (9.7), page 178. Here coefficient matching requires

$$a'(t)/a(t) = t \quad \text{and} \quad a(t)b(t) = e^{x^2/2}.$$

The first equation and $a(0) = 1$ gives us $a(t) = e^{t^2/2}$ so $b(t) = 1$. The representation (9.7) then gives $X_t = e^{t^2/2}\{1 + B_t\}$. One can check by Itô's formula that this does solve the SDE (9.37).

Solution for Problem 9.2. For the same reasons noted in the solution of Problem 9.1, this is a candidate for coefficient matching with the product process (9.7). We need $a'(t)/a(t) = -2/(1-t)$ and $a(t)b(t) = \{2t(1-t)\}^{1/2}$, and these give us $a(t) = (1-t)^2$ and $b(t) = (2t)^{1/2}(1-t)^{3/2}$, so

$$X_t = (1-t)^2 \int_0^t (2u)^{1/2}(1-u)^{3/2}\, dB_u, \quad 0 \le t \le 1.$$

From the Itô isometry we find for $s < t$ that $\mathrm{Cov}(X_s, X_t) = s^2(1-t)^2$, which is the square of the covariance function for the Brownian bridge.

Solution for Problem 9.3. If we substitute $X_t = f(B_t)$, then coefficient matching versus Itô's formula requires the two relations

$$f'(B_t) = \sqrt{1 + f^2(B_t)} \quad \text{and} \quad \frac{1}{2}f''(B_t) = \frac{1}{2}f(B_t). \tag{15.56}$$

For the second ODE to hold for all ω we need $f''(x) = f(x)$ for all x, so we must have $f(x) = Ae^x + Be^{-x}$. Squaring the first ODE then gives us

$$(Ae^x - Be^{-x})^2 = 1 + (Ae^x + Be^{-x})^2 \quad \text{or} \quad 4AB = -1.$$

The boundary condition $f(0) = 0$ requires us to have $A + B = 0$, so we find $A = -B = \pm 1/2$. This gives us $f(x) = \pm\sinh(x)$, but the first condition of