Stochastic Calculus and Financial Applications
Final Take Home Exam (Fall 2006)
SOLUTIONS

INSTRUCTIONS. You may consult any books or articles that you find useful. If you use a result that is not from our text, attach a copy of the relevant pages from your source. You may use any software, including the internet, Mathematica, Maple, R, S-Plus, MatLab, etc. **Attach any Mathematica (or similar) code that you use.**

- You may **NOT** consult with any other person about these problems. If you have a question, even one that is just about the meaning of a question, please contact me directly rather than consult with a fellow student.
- I may post "bug reports" or clarifications on our web page, and you should regularly check for these.
- You should strive to make your answers as **clear and complete** as possible. Neatness counts — especially of thought, but even of handwriting. If I can't read it, I can't grade it.
- Never, ever, write down anything that you know — or even vaguely suspect — to be false. If you understand that your argument is incomplete or only heuristic, this may be fine, but it must be properly labeled as incomplete or heuristic.
- Don't skip steps. If I can't go from line n to line $n+1$ in my head, something is missing. If you use Mathematica or a fact from a table, please say so **and document it.** Otherwise, I stare and stare at line n wondering how you got to $n+1$ in your head while I can't.
- Use anything from anyplace, but do not steal. If you make use of an argument from some source, give credit to the source. If you find the complete (and correct!) solution to a problem in a book or on the internet, just print out the pages and attach them. You will get full credit.
- Write on only one side of a page. Use decent, homogeneous, high quality paper. No napkins, hoagie sacks, *Indian Chief Tablets*, etc.
- Begin each new solution on a new page.
- Arrange your solutions in the natural numerical order. If you do not do problem K, then include a self-standing page that says "Problem K was not done."
- Staple your pages neatly with a high-quality stapler with appropriate length and weight to do a clean and secure job.

- **As discussed in class, you MUST use and complete the cover page given at the website. Self-evaluation is hugely valuable.**

GENERAL ADVICE

1. Avoid the temptation to just write down things that you think are relevant even though these "things" do not add up to an honest solution. Such near-nonsense lists just waste everyone's time.
2. If you can explain **clearly** something that you tried that did not work, this sometimes is worth a few points. Please do not abuse this offer. With experience, one learns that many sensible ideas do not work. Almost by definition, this is what separates the trivial from the non-trivial.
3. Try to keep in mind that a good problem requires that one "overcome some objection." What distinguishes a problem from an exercise is that in a good problem a routine plan does not work. The whole point is to go past the place where routine ideas take you. Still, don't shy away from the obvious; many of the "problems" here are exercises.
4. If you do something **extra** that is valid, you can get "bonus" points. These special rewards cannot be determined in advance. They are usual small, but they can be substantial — and they do add up.
5. The most common source of bonus points is for saying something particularly well. Clear, well-organize, solutions are gems. They deserve to be acknowledged.

THE BIG PICTURE

Almost certainly these instructions will seem to be overly detailed to you. It is true that they are detailed, but they evolved case by case. Each rule deals with some previous mess or misunderstanding. When you start teaching (and grading) I encourage you to follow this example. There is no dishonor in a creative eccentricity or two.

There is a final — more important — motivation for this long list of rules and suggestions. **Detailed instructions provide honest coaching for excellence.** This is the principal benefit. Nevertheless, I hope that no one minds that these rules will also save many hours of everyone's time.

Due Date and Place: The exam **with its completed self-evaluation cover sheet** is due in my office JMHH 447 at 11am on Tuesday December 19.

Problems for Everyone

Problem 1: A Construction of Brownian Motion on $[0, \infty)$

Given that a Brownian motion $\{B(t) : 0 \le t \le 1\}$ has been defined on the unit interval, define an new process on all of $0 \le t < \infty$ by setting

$$\tilde{B}(t) \equiv (1+t)\left\{ B\left(\frac{t}{1+t}\right) - \frac{tB(1)}{1+t} \right\}.$$

Confirm that this process is indeed a Brownian motion on $[0, \infty)$. Although this construction of Brownian motion on $[0, \infty)$ has fewer seams than the one given in the text, it is probably less obvious to most people.

Solution 1: A Construction of Brownian Motion on $[0, \infty)$

It is immediate that $\tilde{B}(t) = 0$ and that $\tilde{B}(t)$ is a Gaussian process with continuous paths. We also have $E(\tilde{B}(t)) = 0$, so we just need to check that $E(\tilde{B}(s)\tilde{B}(t)) = s$ whenever $0 \le s \le t$. One does this simply by substituting the definition of $\tilde{B}$, multiplying out terms, and using $E(B(s')B(t')) = \min(s', t')$ for s' and t' in $[0, 1]$.

Problem 2: Intuition about Volatility from Three Perspectives

Consider a European call option with the current stock price equal to the current strike price. These are commonly called at-the-money options, though there are more sophisticated definitions of "at-the-money."

(a) Now suppose that volatility is zero. If you make the usual Black-Scholes assumptions, explain how one can guess the value of the option by direct reasoning without using either the Black-Scholes formula or the Black-Scholes PDE.

(b) Now take the Black-Scholes PDE and let $\sigma = 0$. Does the value that you obtained in Part (a) solve that simplified PDE and its terminal condition?

(c) Finally, take the Black-Scholes formula, and calculate what you get when $\sigma \to 0$. Be sure to handle any indeterminate expressions honestly.

Incidentally, Richard Feynman often said that you understand an equation when "without solving the equation you can still say how the solutions behave." Feynman's criterion is worth keeping in mind anytime one meets an equation — new or old.

Solution 2: Intuition about Volatility from Three Perspectives

For Part (a) one could reason as follows. "If σ is zero, then at time T the stock will be worth $S_t e^{\mu\tau}$, but if $\mu \ne r$ then I would have an arbitrage possibility. Hence, letting σ be zero, forces $\mu = r$. Hence, at time T my stock will be priced $e^{\tau r} S_t$, and an option owner can buy it for K. Hence at time T, the option owner wins precisely $w = e^{\tau r} S_t - K$. Right now, that payout has value $e^{-\tau r}w = S_t - Ke^{-\tau r}$. Accordingly, this must be the value of the option." To be sure, this is informal, but it is quite sensible.

When $\sigma = 0$ the Black-Scholes PDE becomes $f_t = -rxf_x + rf$, and it is trivial to check that $f(t,x) = x - Ke^{-\tau r}$ does solve this equation.

For Part (c) one just notes that both D_+ and D_- are asymptotic to $r\sqrt{\tau}/\sigma$ and thus go to infinity as $\sigma \to 0$. This gives $\Phi(D_+) \to 1$ and $\Phi(D_-) \to 1$ so the Black-Scholes formula becomes $S - e^{-r\tau}K$. This gives a second confirmation of the intuitive reasoning in Part (a).

PROBLEM 3: IS IT BROWNIAN MOTION?

(a) Consider the process, $X_t = B_{2t} - B_t$, $0 \le t < \infty$. Is it a Gaussian process? Can you find the mean and variance? Is it Brownian motion?

(b) Let X_t and Y_t be independent Brownian motions. Let $Z_t = (X_t + Y_t)/\sqrt{2}$. Is it a Gaussian process? Can you find the mean and variance? Is it Brownian motion?

SOLUTION 3: IS IT BROWNIAN MOTION?

(a) Yes, X_t is a Gaussian process. We must check that for each $(t_1, t_2, ..., t_n)$ that $(X_{t_1}, X_{t_2}, ..., X_{t_n})$ is multivariate Gaussian. That is, we need to show that for any θ_1, θ_2,.., θ_n the sum $S = \theta_1 X_{t_1} + \theta_2 X_{t_2} + \cdots + \theta_n X_{t_n}$ is univariate Gaussian. Since S can be written as a linear combination of values of Brownian motion at various times and since Brownian motion is a Gaussian process, we see the S is indeed Gaussian. Next, we trivially have $E(X_t) = 0$ and $\text{Var}(X_t) = t$ for all t. Finally, since $\text{Cov}(X_s, X_t)$ equals

$$E((B_{2s} - B_s)(B_{2t} - B_t) = E(B_{2s}B_{2t} - B_{2s}B_t - B_sB_{2t} + B_sB_t),$$

we see $\text{Cov}(X_s, X_t) = 3\min(s,t) - \min(2s,t) - \min(s,2t)$. For $s = 2$, $t = 3$ this works out to 1, but the corresponding covariance for Brownian motion is 2. Hence, $\{X_t\}$ is not Brownian motion.

(b) Yes, it's immediate that Z_t is a Gaussian process with mean zero and variance t. Moreover, for $s \le t$ we have

$$\begin{aligned} E(Z_sZ_t) &= (1/2)E(X_sX_t + X_sY_t + Y_sX_t + Y_sY_t) \\ &= (1/2)(s + 0 + 0 + s) = s. \end{aligned}$$

Thus, $\text{Cov}(Z_s, Z_t) = \min(s,t)$, so we see $\{Z_t\}$ is a standard Brownian motion.

PROBLEM 4: INTEGRAND AND INTEGRAL SIZE

Suppose that $\phi(\omega, t)$ is a stochastic integrand for which there is a constant C such that

$$E(\phi^2(\omega, t)) \le Ct^p \quad \text{for all } t \ge 0. \tag{1}$$

Show that the stochastic integral

$$X_t = \int_0^t \phi(\omega, s)\, dB_s$$

satisfies the bound

$$E(|X_t|) \le \left(\frac{C}{p+1}\right)^{1/2} t^{(p+1)/2}.$$

Solution 4: Integrand and Integral Size

We combine the Cauchy-Schwarz inequality with the Itô isometry. Specifically, we have

$$E(|X_t|) \le E(X_t^2)^{1/2} = \left\{ E \int_0^t \phi^2(\omega, s)\, ds \right\}^{1/2}.$$

Now, change the order of integration and expectation and replace the expectation $E(\phi^2(\omega, s)$ by the bound Cs^p.

Problem 5: A Greek Bound on the Call Price

Let $f(t, S_t)$ denotes the time t arbitrage-free price of a call option under the usual Black–Scholes model, and explain why one has the bound

$$f(t, S_t) \le S_t \Delta_t + \frac{\sigma^2}{2r} S_t^2 \Gamma_t,$$

where, as usual, we have $\Delta_t = f_x(t, S_t)$ and $\Gamma_t = f_{xx} t, S_t)$.

Solution 5: A Greek Bound on the Call Price

This is almost free. The Black–Scholes PDE can be written as

$$rf(t, S_t) = f_t(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 f_{xx}(t, S_t) + rS_t f_x(t, S_t).$$

The key observation is that $f_t(t, S_t) \le 0$, which is financially obvious since, if the only thing that changes is time, then the option must become steadily less valuable. After guessing this relation, one gives a formal derivation by differentiating the Black-Scholes formula an collecting terms.

Problem 6: A Text Book Blooper

A recently published text asserts that for each fixed $\gamma > 0$ the collection of random variables

$$\mathcal{C} = \{X_t = \exp(\gamma B_t - \gamma^2 t/2) : 0 \le t < \infty\}$$

is uniformly integrable. Prove that this statement is false.

Curiously enough, $\mathcal{C}$ is almost a "poster child" for a collection that fails to be uniformly integrable. Dozens of papers offer conditions on a stopping time τ that suffice to give $E[\exp(B_\tau - \tau/2)] = 1$, and all of these would be irrelevant if $\mathcal{C}$ were a uniformly integrable collection.

Solution 6: A Text Book Blooper

Take any $\gamma > 0$. By the law of large numbers for Brownian motion, the exponent

$$\gamma B_t - \gamma^2 t/2 = t\gamma(B_t/t - \gamma/2)$$

diverges to minus infinity as $t \to \infty$. Hence for $X_t = e^{\gamma B_t - \gamma^2 t/2}$ we see that

$$\lim_{t\to\infty} X_t = 0 \text{ with probability one, yet } E(X_t) = 1 \text{ for all } t.$$

On the other hand, for any uniformly integrable family $\{Y_t\}$ such that Y_t converges to Y in probability (or with probability one) we have $E(Y_t) \to E(Y)$. Indeed, this property largely accounts for the usefulness of the notion of uniform integrability.

Problem 7: A Hitting Time Identity

Let $\tau = \min\{t : B_t = 1\}$, and show that we have the nice identity

$$E[\exp(-\tau/2)] = \frac{1}{e}.$$

We know the density of τ, but direct integration would not be the most pleasing way to obtain this expectation. What's wanted here is a simple martingale argument. Incidentally, this formula adds something to our intuition about τ. We know that $E(\tau) = \infty$, but this new formula says that "properly discounted" τ is "around" 2.

Solution 7: A Hitting Time Identity We know that $X_t = \exp(B_t - t/2)$ is a martingale, so by Doob's stopping time theorem, so is $X_{t\wedge\tau}$. This gives us $E(X_{t\wedge\tau}) = 1$ for all $t \geq 0$. Since $X_{t\wedge\tau}$ is nonnegative and bounded by e, the fact that $P(\tau < \infty) = 1$ and the DCT then gives us $E(X_\tau) = 1$. Since $X_\tau = \exp(1 - \tau/2)$, taking the expectation gives us our identity.

Problem 8: A Martingale and an Integral

Show that if g is a continuously differentiable function, then the process

$$M_t = g(t)B_t - \int_0^t g'(s)B_s\,ds$$

is a martingale. Next, show that for $\tau = \min\{t : B_t = A \text{ or } B_t = -B\}$ one has

$$E[\sin(\tau)B_\tau] = E\left(\int_0^\tau \cos(s)B_s\,ds\right). \tag{2}$$

Solution 8: A Martingale and an Integral

For $f(t,x) \equiv g(t)x$ we have $f \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R})$, so Itô's formula can be applied. Since $f_x = g$, $f_{xx} = 0$, and $f_t = xg'$ we then find

$$g(t)B_t = \int_0^t g'(s)B_s\,ds + \int_0^t g(s)\,dB_s.$$

Now, the second integral equals M_t, and, since g is bounded on any interval $[0,T]$, this integral is a martingale. Finally, to prove the identity (2), we first note by Doob's stopping time theorem that $M_{t\wedge\tau}$ is also a martingale, so we have $E(M_{t\wedge\tau}) = 0$. In other words, we have

$$E[B_{\tau\wedge t}\sin(\tau\wedge t)] = E\left(\int_0^{\tau\wedge t}\cos(s)B_s\,ds\right).$$

Since $|B_t| \le \max(A,B)$ for $t \le \tau$, the DCT permits us to take limits inside both of these expectations. Doing so completes the proof.

Problem 9: All that Glitters is not Brownian Motion

Suppose that X_t and Y_t are two continuous processes such that for each pair of constants α and β with $\alpha^2+\beta^2 = 1$ the process $Z_t = \alpha X_t + \beta Y_t$ is a standard Brownian motion. First, show that for each t the random variables X_t and Y_t are independent. Second, show by example that the processes $\{X_t\}$ and $\{Y_t\}$ need not be independent.

Big Hint: OK, for part two, I'll show you the example, but you still have to check that it works. Consider independent Brownian motions $B_1(t)$ and $B_2(t)$, and then set $X_t = B_1(2t/3) - B_2(t/3)$ and $Y_t = B_1(t/3) + B_2(2t/3)$.

Solution 9: All that Glitters is not Brownian Motion

By taking $\alpha = 1$ and $\beta = 0$ we see that X_t is a standard Brownian motion and taking $\alpha = 0$ and $\beta = 1$ shows that Y_t is a standard Brownian motion. The issue is to prove independence. By our hypothesis and the definition of the multivariate Gaussian distribution, we see that for each s and t the pair (X_s, Y_t) is bivariate Gaussian; thus to show independence, we just need to show $E(X_sY_t) = 0$. Take $\alpha = \beta = 1/\sqrt{2}$. We have $\mathrm{Var}[(X_s+Y_t)/\sqrt{2}] = t$ from our hypothesis, so we have

$$t = \frac{1}{2}\mathrm{Var}(X_t) + \frac{1}{2}\mathrm{Var}(Y_t) + \mathrm{Cov}(X_t, Y_t).$$

Since $\mathrm{Var}(X_t) = \mathrm{Var}(Y_t) = t$ this gives $\mathrm{Cov}(X_t,Y_t) = 0$. Since (X_s, Y_t) is bivariate Gaussian this implies that X_t and Y_t are independent.

For the example, it is immediate that $E(X_t^2) = t$, $E(X_t^2) = t$, and one easily checks $E(X_tY_t) = 0$. Since (X_t,Y_t) bivariate Gaussian, we see that Z_t is Gaussian, mean zero, and $\mathrm{Var}(Z_t) = t$. The definition also shows that Z_t has independent increments, so Z_t is a Brownian motion. To see that the processes

$\{X_t\}$ and $\{Y_t\}$ are dependent, just compute $E(X_s Y_t)$ for some $0 < s < t$. Taking $s = 1$ and $t = 2$ gives $1/3$, so the processes $\{X_t\}$ and $\{Y_t\}$ are not independent.

[Clyde D. Hansen, Jr. (1985) "A Spurious Brownian Motion," Proceedings of the American Mathematical Society, **93** (2), 350.]

Problems with a Challenge, but Still Solidly in Range

PROBLEM 10: THE BLACK–SCHOLES "LOWER CASE" TRANSFORMATION

Take the Black–Scholes formula, replace Φ by ϕ, and simplify. I mean *really simplify*!. What do you get? I promise that you will remember the stunning answer for the rest of your life. It's also financially informative. Show your work!

SOLUTION 10: THE BLACK–SCHOLES "LOWER CASE" TRANSFORMATION

The answer is zero. You can give a proof that parallels our proof that $\Delta = \Phi(D_+)$, but, with afterthought, there is a more elegant arrangement. First we note

$$\frac{\phi(D_+)}{\phi(D_-)} = \exp\left(-\frac{1}{2}(D_+^2 - D_-^2)\right) = \exp\left(-\frac{1}{2}(D_+ + D_-)(D_+ - D_-)\right),$$

and, as before, we have the two relations

$$D_+ + D_- = 2\frac{\log(S/K) + r\tau}{\sigma\sqrt{\tau}} \quad \text{and} \quad D_+ - D_- = \frac{\sigma\tau}{\sigma^2\tau} = \sigma\sqrt{\tau}.$$

From these we see

$$\frac{1}{2}(D_+ + D_-)(D_+ - D_-) = \log(S/K) + r\tau = \log\left(\frac{S}{Ke^{-r\tau}}\right),$$

and hence one finds the lovely — and conceptually informative — identity

$$\frac{\phi(D_+)}{\phi(D_-)} = \frac{Ke^{-r\tau}}{S}. \tag{3}$$

When the identity (3) is rewritten without fractions, it tells us that changing Φ to lower case ϕ in the Black-Scholes formula gives us zero. Moreover, the formula (3) offers what is perhaps the best way to make intuitive sense of the ubiquitous factors D_- and D_+. Among other things it suggests that "moneyness" can be viewed as a kind of likelihood ratio.

PROBLEM 11: TEXAS CHAIN RULE MASSACRE

Suppose that $f(t,x)$ solves the Black-Scholes PDE with the usual terminal condition $f(T,x) = (x-K)_+$. Consider the new variable y and the new function $g(t,y)$ that are defined by the relations:

$$y = e^{r(T-t)}x \quad \text{and} \quad g(t,y) = e^{r(T-t)}f(t,x).$$

Show that g satisfies the more pleasant terminal value PDE

$$\frac{\partial g}{\partial t} + \frac{1}{2}\sigma^2 y^2 \frac{\partial^2 g}{\partial y^2} = 0 \quad \text{and} \quad g(T, y) = (y - K)_+. \tag{4}$$

This certainly a nice simplification of the Black-Scholes equation, and it suggests that the new variables are "better" variables. With time on our hands, we could see what other simplifications might be achieved with using y and g in place of x and f.

HINT: This is only a chain rule exercise, but it is easily messed up. If you make a mistake in your calculations and still arrive at the target formula, you may have engaged in a willful fiction — which is not polite.

To succeed here is easy, if you start out right. The right way to start is to first write the "old stuff" as a function of the "new stuff." You can then do straightforward chain rule calculations of the derivatives associated with the "old stuff," plug these into the "old equation," and get the "new equation." Less systematic approaches are often fought both with error and irrelevant algebra.

SOLUTION 11: TEXAS CHAIN RULE MASSACRE

We have $x = e^{-r(T-t)}y$ and $f(t,x) = e^{-r(T-t)}g(t,y)$. We can't forget that $\partial y/\partial x = e^{r(T-t)}$ and

$$\partial y/\partial t = -e^{-r(T-t)}xr = -ry.$$

Now we just work out the various derivatives f using the chain rule:

$$\begin{aligned}
f_t(t,x) &= re^{-r(T-t)}g(t,y) + e^{-r(T-t)}g_t(t,y) - e^{-r(T-t)}(-ry)g_y(t,y) \\
f_x(t,x) &= e^{-r(T-t)}g_y(t,y)(\partial y/\partial x) = g_y(t,y) \\
f_{xx}(t,x) &= g_{yy}(t,y)(\partial y/\partial x) = e^{r(T-t)}g_{yy}(t,y)
\end{aligned}$$

Substitution into the Black–Scholes PDE $f_t = -(1/2)x^2\sigma^2 f_{xx} - rxf_x + rf$ now gives

$$\begin{aligned}
&re^{-r(T-t)}g(t,y) + e^{-r(T-t)}g_t(t,y) - e^{-r(T-t)}(-ry)g_y(t,y) = \\
&-(1/2)(e^{-r(T-t)}y)^2\sigma^2 e^{r(T-t)}g_{yy}(t,y) - re^{-r(T-t)}yg_y(t,y) + re^{-r(T-t)}g(t,y).
\end{aligned}$$

Direct cancelations now give use the simple PDE (4).

Incidentally, y and g are sometimes called the "forward" stock price and option price, although this terminology can collide with other uses of the word "forward." One mathematical motivation for using a change of variables of the form $y = A(t)x$ and $g(t,y) = A(t)f(t,x)$ is that the relation $f_x(t,x) = g_y(t,y)$ is baked into the cake. Also, with $A(t) = e^{r(T-t)}$ we have $A(T) = 1$ so the boundary condition is unchanged.

PROBLEM 12: ONE OF TWO LINES

Consider the standard Brownian motion $Z_t = (X_t, Y_t)$ in $\mathbb{R}^2$, and let τ denote the first time that either $Y_t = aX_t + b$ or $Y_t = aX_t - b$. That is, we let τ denote the first time that Z_t hits the boundary of the strip S defined by the parallel lines $y = ax + b$ and $y = ax - b$. Show that one has

$$E(\tau) = b^2/(1 + a^2),$$

then pose — and prove — a three dimensional generalization.

Solution: One of Two Lines Solution This is almost a freebie. The key observation is that the process $B_t = (Y_t - aX_t)/\sqrt{1+a^2}$ is a standard Brownian motion, and τ is equal to the first time that B_t hits $A = b/\sqrt{1+a^2}$ or hits $-B = -b/\sqrt{1+a^2}$. We've known forever that $E(\tau) = AB$ and here $AB = b^2/(1+a^2)$.

This method easily generalizes to give the expected time for a standard Brownian motion (X_t, Y_t, Z_t) in $\mathbb{R}^3$ to hit the boundary of the slab S determined by $z = ax + by + c$ and $z = ax + by - d$ where $c > 0$ and $d > 0$. This time we have the standard 1-dimensional Brownian motion

$$B_t = (Z_t - aX_t - bY_t)/\sqrt{1 + a^2 + b^2}$$

so upon setting $A = c/\sqrt{1+a^2+b^2}$ and $B = d/\sqrt{1+a^2+b^2}$ we find

$$E(\tau) = AB = cd/(1 + a^2 + b^2).$$

Problem 13: BM without Assuming Independence

Let $\{X_t\}$ denote a mean zero Gaussian process and let $\Delta(s,t) = X_t - X_s$ denote the increment in the process from time s to time t. Assume that for each choice of four times $0 \le t_1 \le t_2 \le t_3 \le t_4$ the variance of the sum $S = \Delta(t_1,t_2) + \Delta(t_2,t_3) + \Delta(t_3,t_4)$ is equal to $t_4 - t_1$. Show that $\{X_t\}$ is Brownian motion. The point here is that our assumption on S is enough to show that $\{X_t\}$ has independent increments.

Solution 13: BM without Assuming Independence

First observe that by taking $t_1 = s$ and $t = t_2 = t_3 = t_4$ we see that $E(\Delta^2(s,t) = t - s$ for all $s \le t$. Next, since we have

$$(\Delta(t_1,t_2) + \Delta(t_2,t_3))^2 = \Delta^2(t_1,t_2),$$

we can expand and take expectations to get

$$t_2 - t_1 + 2E(\Delta(t_1,t_2)\Delta(t_2,t_3)) + t_3 - t_2 = t_3 - t_1.$$

Cancelation shows $E(\Delta(t_1,t_2)\Delta(t_2,t_3)) = 0$, so we see that adjacent increments of $\{X_t\}$ are independent. Now we use this fact together with our full assumption about S. Specifically, squaring S gives

$$S^2 = \Delta^2(t_1,t_2) + \Delta^2(t_2,t_3) + \Delta^2(t_3,t_4) \\ + 2\Delta(t_1,t_2)\Delta(t_2,t_3) + 2\Delta(t_1,t_2)\Delta(t_3,t_4) + 2\Delta(t_2,t_3)\Delta(t_3,t_4).$$

Hence, when we take expectations, we get

$$t_4 - t_1 = t_4 - t_1 + 2E(\Delta(t_1,t_2)\Delta(t_3,t_4)),$$

and cancelation shows $\Delta(t_1,t_2)$ and $\Delta(t_3,t_4)$ are uncorrelated. Since these are two arbitrary time-disjoint increments, we see that the Gaussian process $\{X_t\}$ has independent increments.

Source: Rényi, A. (1967). Remarks on the Poisson Process. *Studia Scientiarum Mathematicarum Hungarica*, **2**, 119–123.

Problem 14: A Stopping Time Inequality

Prove that for any stopping time τ and any $t \geq 0$ one has the inequality

$$E\left(\exp(\frac{1}{2}B_{t\wedge\tau})\right) \leq \{E(\exp(\tau/2))\}^{1/2}. \tag{5}$$

Hint: You'll want to get one of your favorite martingales into the game, and, when a square root is in sight, it never hurts to consider Cauchy-Schwarz.

Solution 14: A Stopping Time Inequality

Since $\exp(B_t - t/2)$ is a martingale, Cauchy-Schwarz and Doob's stopping time theorem give us

$$\begin{aligned} E\left(\exp(\frac{1}{2}B_{t\wedge\tau})\right) &= E\left(\exp(\frac{1}{2}B_{t\wedge\tau} - \frac{1}{4}\tau\wedge t)\exp(\frac{1}{4}\tau\wedge t)\right) \\ &\leq \left\{E\left(\exp(B_{t\wedge\tau} - \frac{1}{2}\tau\wedge t)\right)\right\}^{1/2}\left(E(\exp(\frac{1}{2}\tau\wedge t)\right)^{1/2} \\ &= \left(E(\exp(\frac{1}{2}\tau\wedge t)\right)^{1/2} \leq \left(E(\exp(\frac{1}{2}\tau)\right)^{1/2}. \end{aligned}$$

Remark: This shows that Kazamaki's condition is weaker than Novikov's condition. I learned this from "On Criteria for the Uniform Integrability of Brownian Stochastic Exponentials" by A.S. Cherny and A.N. Shiryaev (unpublished manuscript), but the result probably goes back to Kazamaki.

Problem 15: BM and a Stochastic Integral

Consider the process

$$X_t = \int_0^t \text{sign}(B_s)\, dB_s$$

where $\text{sign}(x) = 1$ for $x \geq 0$ and $\text{sign}(x) = -1$ for $x < 0$. Use Levy's theorem to check that X_t is again a Brownian motion. Next, confirm that the two processes X_t and B_t are uncorrelated. Finally show that

$$E(X_t B_t^2) = 2^{5/2} t^{3/2} / 3\sqrt{\pi}, \tag{6}$$

and explain why this implies that the processes X_t and B_t are not independent — despite being uncorrelated and Gaussian processes.

SOLUTION 15: BM AND A STOCHASTIC INTEGRAL

Since the integrand $\text{sign}(B_s)$ is bounded, the stochastic integral that defines X_t is a martingale, and its quadratic variation is given by

$$\langle X \rangle_t = \int_0^t \text{sign}^2(B_s)\, ds = \int_0^t 1\, ds = t.$$

Hence, X_t is a continuous martingale with quadratic variation t, so by Lévy's characterization we see that X_t is a Brownian motion. Next, since both X_t and B_t have mean zero and both are in $\mathcal{H}^2$, their covariance is given by the polarized Itô Isometry

$$E(B_t X_t) = E\left(\int_0^t dB_s \int_0^t \text{sign}(B_s)\, dB_s \right) = E\left(\int_0^t 1 \cdot \text{sign}(B_s)\, ds \right).$$

By symmetry, $E(\text{sign}(B_s)) = 0$ for each $s \neq 0$, so, when we take the expectation inside the integral, we get $E(B_t X_t) = 0$. Finally, to compute $E(X_t B_t^2)$ we still want to use the Itô Isometry, so we replace B_t^2 by its Itô integral representation. We then compute with the polarized Itô Isometry:

$$\begin{aligned} E(X_t B_t^2) &= E\left(\int_0^t \text{sign}(B_s)\, dB_s \left(2 \int_0^t B_s\, dB_s - t \right) \right) \\ &= 2E\left(\int_0^t \text{sign}(B_s) B_s\, ds \right) = 2 \int_0^t E(|B_s|)\, ds. \end{aligned}$$

Since $E(|B_s|) = \sqrt{2s}/\sqrt{\pi}$, our target formula (6) follows by integration.

PROBLEM 16: BM AND A STOPPING TIME

Show that for any bounded stopping time τ for the Brownian filtration, one has the bound

$$E(\tau^2) \leq c E(B_\tau^4) \quad \text{where} \quad c = (5 + 2\sqrt{6})/3. \tag{7}$$

Moreover, give an example that shows that this bound can fail for an unbounded stopping time.

Hint: In your investigation you may want to exploit the observation that for any x and y and any $\lambda > 0$ one has $2xy \leq \lambda x^2 + y^2/\lambda$. This is a common device for eliminating an awkward product. When the time is right, you then optimize over the free parameter λ.

Solution 16: BM and a Stopping Time

It is natural to expect that we will need a martingale that contains the summands B_t^4 and t^2. One that we have used before is $M_t = B_t^4 - 6tB_t^2 + 3t^2$, so we give it a try. Doob's stopping time theorem gives us $E(M_\tau) = 0$, and this gives us the identity

$$E(\tau^2) = 2E(\tau B_\tau^2) - \frac{1}{3}E(B_\tau^4).$$

Now, using the bound $2\tau B_\tau^2 \leq \lambda\tau^2 + B_\tau^4/\lambda$, we find that for all $0 < \lambda < 1$ that one has the bound

$$E(\tau^2) \leq \frac{3-\lambda}{2\lambda(1-\lambda)}E(B_\tau^4).$$

Optimizing over λ gives $\lambda = 3 - \sqrt{6}$ and substituting this choice in the bound yields the constant c of equation (7). Finally, to see that the boundedness assumption cannot be ignored, consider the $\tau = \min\{t : B_t = 1\}$. We know τ is finite with probability one, but $E(\tau) = \infty$ and $B_\tau \equiv 1$ so the inequality (7) is rudely violated.

Problems that Are Harder but Still NOT Too Hard

Problem 17: Twin Bounds on an Exit Time

Consider a process X_t with $X_0 = 0$ that satisfies $dX_t = \sigma(X_t)\,dB_t$ where $\sigma(x)$ is a smooth function constrained by the constant bounds

$$0 < \sigma_{\text{low}} \leq \sigma(x) \leq \sigma_{\text{high}} < \infty.$$

Let $A > 0$ and $B > 0$ and set $\tau = \min\{t : X_t = A \text{ or } X_t = -B\}$. After first showing that $E(\tau) < \infty$, show more precisely that one has the twin bounds:

$$AB/\sigma_{\text{high}}^2 \leq E(\tau) \leq AB/\sigma_{\text{low}}^2. \tag{8}$$

Here, one should note that if $\sigma_{\text{high}} = \sigma_{\text{low}}$ then these bounds recover an equality that we have met several times before.

Solution 17: Twin Bounds on an Exit Time

To follow the pattern of our other hitting time problems we consider the martingale

$$M_t = X_t^2 - \int_0^t \sigma^2(X_s)\,ds.$$

By Doob's stopping time theorem, we have $E(M_{\tau\wedge t}) = 0$, and this gives us

$$E\int_0^{\tau\wedge t} \sigma^2(X_s)\,ds = E(X_{\tau\wedge t}^2). \tag{9}$$

When we bound $X_{\tau\wedge t}^2$ by $\max(A^2, B^B)$ and use the lower bound on σ in the integral we get the crude bound

$$\sigma_{\text{low}}^2 E(\tau \wedge t) \leq \max(A^2, B^2).$$

Fatou's lemma now gives us $E(\tau) \leq \max(A^2, B^2)/\sigma_{\text{low}}^2 < \infty$, so, in particular, we have $P(\tau < \infty) = 1$.

Given this, we take the limits in the identity (9) using the MCT on the left and the DCT on the right to get our key result

$$E \int_0^\tau \sigma^2(X_s)\, ds = E(X_\tau^2) = AB. \tag{10}$$

Here, the second equality comes from observing that X_t is a continuous martingale. For such processes we always have

$$P(X_\tau = A) = B/(A+B) \quad \text{and} \quad P(X_\tau = -B) = A/(A+B).$$

and these imply $E(X_\tau^2) = AB$. Finally, since we have the trivial bounds,

$$\sigma_{\text{low}}^2 E(\tau) \leq E \int_0^\tau \sigma^2(X_s)\, ds \leq \sigma_{\text{high}}^2 E(\tau), \tag{11}$$

substitution from the key result (10) completes the job. Incidentally, this can also be done in many ways. My first proof was based on aa direct appeal to the Itô isometry, and it was a little longer and a little trickier. You can also give a proof by writing X_t as a time change of Brownian motion.

Problem 18: The Famous Ft. Worth Two Step Option

Price an option in Black-Scholes world that pays you one buck at time T if $S_T \geq K_1$ and pays you two bucks if $S_T \geq K_2$ where $0 \leq K_1 < K_2$ are two given "strike prices." Include the details for your computations, or, if you use Mathematica, provide your code and out-put (all cleaned up, but executable). Actually, this computation is easily done by hand and the answer is not more complicated than the Black-Scholes formula.

As a hint, let me suggest that you use the risk-neutral pricing approach rather than the PDE approach. This should make things pretty straightforward. To get the full experience, be sure to include the details of any integrals or probabilities that you compute.

Solution 18: The Famous Ft. Worth Two Step Option

Consider the arbitrage price $V(S, K, \tau, r, \sigma)$ of an option that pays one buck if $S_T > K$. The Ft. Worth Two Step has arbitrage price

$$V(S, K_1, \tau, r, \sigma) + V(S, K_2, \tau, r, \sigma).$$

Now consider the fundamental pricing formula with $X = \mathbb{I}(S_T \geq K)$ and recall that under the equivalent martingale measure (or risk-neutral measure) the stock price process is a geometric Brownian motion with growth parameter r and volatility parameter σ. If $S_t = S$ and $\tau = T - t$ then under Q we see that S_T has the the same distribution as

$$S \exp\left((r - \sigma^2)\tau + \sigma\sqrt{\tau} Z\right)$$

where Z is a standard normal, i.e. $P_Q(Z \leq x) = \Phi(x)$. Hence, we have

$$\begin{aligned} P_Q(S_T \geq K \mid S_t = S) &= P_Q((r - \sigma^2)\tau + \sigma\sqrt{\tau} Z \geq \log(K/S)) \\ &= P_Q\left(-\sigma\sqrt{\tau} Z \leq \log(S/K) + (r - \sigma^2)\tau\right) \equiv \Phi(D_-), \end{aligned}$$

where in the last line we used the symmetry of the distribution of Z. Hence we have

$$V(S, K, \tau, r, \sigma) = e^{-r\tau}\Phi(D_-),$$

and nothing more is needed.

Envoi

I hope that at least some of these problems are interesting to you. Perhaps one or two may even offer a mild epiphany. They have been created for your enjoyment.

These problems should be fully accessible to everyone. Still, easy or hard, I hope that at least a few will scratch out some higher, more conceptual messages. That is the intention behind their design.

There are a few problems that will provide some challenge to almost anyone, but even if you took a brief “break” from the course, don’t count yourself out. You can still do all of these problems if that is your desire and if you have time to give them an honestly try.

Still, life is short. You should do the problems you want to do and skip the rest. Whatever problems you chose to solve, I promise to read your solutions carefully. I will do my best to understand your ideas.

Good luck to all!