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## Appendix I: Problem Hints and Solutions

### Chapter 1

SOLUTION FOR PROBLEM 1.1. Let  $T_{i,j}$  denote the expected time to go from level  $i$  to level  $j$ , and note by formula (1.16) that  $T_{25,20} = 15$  and  $T_{21,20} = 3$ . check! By first-step analysis we also have

$$T_{20,19} = \frac{1}{10} \cdot 1 + \frac{9}{10} \cdot \{1 + T_{21,20} + T_{20,19}\}$$

so substituting  $T_{21,20} = 3$  and solving gives  $T_{20,19} = 37$ . Similarly, we have

$$T_{19,18} = \frac{1}{3} \cdot 1 + \frac{2}{3} \cdot \{1 + T_{20,19} + T_{19,18}\},$$

so substituting  $T_{20,19} = 37$  and solving gives  $T_{19,18} = 77$ . Finally, one finds

$$T_{25,18} = T_{25,20} + T_{20,19} + T_{19,18} = 15 + 37 + 77 = 129.$$

SOLUTION FOR PROBLEM 1.2. We get (1.26) just by substitution, and we also have

$$N_n = \sum_{k:2k \leq n} \mathbb{I}(S_{2k} = 0) \quad \text{and} \quad E(N_n) = \sum_{k:2k \leq n} P(S_{2k} = 0),$$

so (1.26) and integral comparison give us

$$\sum_{k:2k \leq n} P(S_{2k} = 0) \sim \sum_{1 \leq k \leq n/2} 1/\sqrt{\pi k} \sim \sqrt{2n/\pi} \quad \text{as } n \rightarrow \infty.$$

SOLUTION FOR PROBLEM 1.3. First set  $N_\infty = \lim N_n$  and then note that  $P(N_\infty \geq k) = r^k$  since the event  $\{N_\infty \geq k\}$  entails  $k$  successes of independent events each of which has success probability  $r$ . Now, if it were truly the case that  $0 \leq r < 1$ , then we would have

$$E(N_n) \leq E(N_\infty) = \sum_{k=1}^{\infty} P(N_\infty \geq k) = \sum_{k=1}^{\infty} r^k = \frac{r}{1-r} < \infty,$$

but by Exercise 1.2 we know  $E(N_n) \sim \sqrt{2n/\pi}$ , so we must have  $r = 1$ .

SOLUTION FOR PROBLEM 1.4. First-step analysis gives us

$$P(\tau_0 = 2k) = \frac{1}{2}P(\tau_0 = 2k \mid X_1 = 1) + \frac{1}{2}P(\tau_0 = 2k \mid X_1 = -1),$$

but by symmetry and the identity (1.24) we have

$$P(\tau_0 = 2k \mid X_1 = 1) = P(\tau_0 = 2k \mid X_1 = -1) = \frac{1}{2k-1} \binom{2k}{k} 2^{-2k}$$

from which (1.28) follows. The asymptotic formula (1.29) follows directly from Stirling's formula, and the relation  $E(\tau_0) = \infty$  is also straightforward. With just a little more work, one can use (1.29) to check that  $E(\tau_0^\alpha) < \infty$  for all  $\alpha < \frac{1}{2}$  and that  $E(\tau_0^\alpha) = \infty$  for all  $\alpha \geq \frac{1}{2}$ .

SOLUTION FOR PROBLEM 1.5. To prove the first identity of (1.31), note that for  $k \geq 1$  the event  $\{L_k > 0\}$  cannot occur unless the first step of the random walk is to +1. If the first step is to +1, then  $\{L_k > 0\}$  occurs if and only if the walk hits  $k$  before it hits 0. By (1.2) this occurs with probability  $1/k$ . When we put these two independent requirements together, we see that  $P(L_k > 0) = (1/2)(1/k)$ .

To prove the second identity of (1.31), we consider a time at which the walk hits level  $k$ , and we make two observations. If on its next step the walk goes up, then it is guaranteed to hit level  $k$  at least one more time before it hits level 0. On the other hand, if the walk goes down on the next step after hitting level  $k$ , then by (1.2) the walk will hit level  $k$  a least one more time with probability  $(k-1)/k$ . These observations combine to give us (1.31).

Finally, to prove (1.30), we note that

$$\begin{aligned} P(N_k > j) &= P(N_k > 0)P(N_k > 1 \mid N_k > 0) \cdots P(N_k > j \mid N_k > j-1) \\ &= \frac{1}{2k} \left( \frac{1}{2} + \frac{1}{2} \frac{k-1}{k} \right)^j = \frac{1}{2k} \left( \frac{2k-1}{2k} \right)^j. \end{aligned}$$

If we now sum over  $0 \leq j < \infty$  we get  $E(N_k)$  on the left, while on the right we see that geometric summation gives us exactly 1.

Incidentally, this problem has an elegant generalization to biased random walk where  $p < q$ . If we repeat our argument but use the ruin probability formula (1.13) in place of the formula (1.2) for unbiased ruin probabilities, we discover that  $E(L_k) = (p/q)^k$ . For  $p = q$  this recaptures the formula (1.30), and, in a way, it explains why  $E(L_k)$  does not depend on  $k$  for unbiased walk.

SOLUTION FOR PROBLEM 1.6. Let  $N_{(1,1)(\alpha+\beta,\alpha-\beta)}$  denote the total number of random walk paths from  $(1, 1)$  to  $(\alpha + \beta, \alpha - \beta)$  and let

$$N_{(1,1)(\alpha+\beta,\alpha-\beta)}^{\text{DoTouch}} \quad \text{and} \quad N_{(1,1)(\alpha+\beta,\alpha-\beta)}^{\text{DoNotTouch}}$$

denote the corresponding number of paths that respectively *do* and *do not* touch the axis. By the reflection principle and path counting we find

$$\begin{aligned} N_{(1,1)(\alpha+\beta,\alpha-\beta)}^{\text{DoNotTouch}} &= N_{(1,1)(\alpha+\beta,\alpha-\beta)} - N_{(1,1)(\alpha+\beta,\alpha-\beta)}^{\text{DoTouch}} \\ &= N_{(1,1)(\alpha+\beta,\alpha-\beta)} - N_{(1,-1)(\alpha+\beta,\alpha-\beta)} \\ &= \binom{\alpha+\beta-1}{\alpha-1} - \binom{\alpha+\beta-1}{\alpha} = \frac{\alpha-\beta}{\alpha+\beta} \binom{\alpha+\beta}{\alpha}, \end{aligned}$$

so the probability that  $A$  leads throughout the counting process is

$$N_{(1,1)(\alpha+\beta,\alpha-\beta)}^{\text{DoNotTouch}} / N_{(0,0)(\alpha+\beta,\alpha-\beta)} = N_{(1,1)(\alpha+\beta,\alpha-\beta)}^{\text{DoNotTouch}} / \binom{\alpha+\beta}{\alpha} = \frac{\alpha-\beta}{\alpha+\beta}.$$

### Chapter 2

SOLUTION FOR PROBLEM 2.1. The solutions of the equation

$$1 = E(y^{X_1}) = 0.52y^{-1} + 0.45y + 0.03y^2$$

are  $y = 1$ ,  $y = 1.01849$ , and  $y = -17.01849$ , and for any one of these  $M_n = y^{S_n}$  is a martingale. Since  $y = 1$  gives a trivial martingale and since  $y = -17.01849$  becomes unruly when raised to a high power, we take  $y = 1.01849$  to define  $M_n$ . We then argue as before that  $E(M_\tau) = 1$ , and this gives us

$$1 = y^{100}P(S_\tau = 100) + y^{101}P(S_\tau = 101) + y^{-100}P(S_\tau = -100).$$

Now, if we let  $p = P(S_\tau = 100) + P(S_\tau = 101)$ , the fact that  $y > 1$  gives us

$$1 < y^{101}p + y^{-100}(1-p) \quad \text{and} \quad 1 > y^{100}p + y^{-100}(1-p).$$

Solving for  $p$  and substituting for  $y$  gives

$$\frac{1 - y^{-100}}{y^{101} - y^{-100}} < p < \frac{1 - y^{-100}}{y^{100} - y^{-100}} \quad \text{or} \quad 0.1353 < p < 0.1379,$$

so  $p$  is determined within an error of  $3 \times 10^{-3}$ .

Incidentally, by taking advantage of high-precision arithmetic (say as provided by *Mathematica*), one can use the *pair* of martingales determined by  $y = 1.01849$  and  $y = -17.01849$  to obtain a system of two equations in two unknowns which can be solved exactly for all of the values  $P(S_\tau = 100)$ ,  $P(S_\tau = 101)$ , and  $P(S_\tau = -100)$ . This pleasing trick often helps, and it suggests a general principle: Two martingales can be better than one!

The full answer?

SOLUTION FOR PROBLEM 2.2. Since  $\{A_n\}$  is bounded and adapted, we see that  $\widetilde{M}_n$  is in  $\mathcal{F}_n$  and integrable for each  $n \geq 0$ . To check the martingale identity we note

$$\begin{aligned} E(\widetilde{M}_n | \mathcal{F}_{n-1}) &= E(\widetilde{M}_{n-1} + A_n(M_n - M_{n-1}) | \mathcal{F}_{n-1}) \\ &= \widetilde{M}_{n-1} + A_n E(M_n - M_{n-1} | \mathcal{F}_{n-1}) \geq \widetilde{M}_{n-1} \end{aligned}$$

since  $A_n E(M_n - M_{n-1} | \mathcal{F}_{n-1})$  is nonnegative.

SOLUTION FOR PROBLEM 2.3. First we notice that we can assume without loss of generality that  $M_0 = 0$  since

$$E[M_\nu] \leq E[M_\tau] \iff E[M_\nu - M_0] \leq E[M_\tau - M_0].$$

Now we apply the result of Problem 2.2 with the choice

$$A_k = \mathbb{I}(\nu < k \leq \tau) = 1 - \mathbb{I}(\tau \leq k - 1) - \mathbb{I}(\nu \leq k - 1). \quad (15.39)$$

By the first equation of (15.39), we see that  $A_k$  is nonnegative and bounded. By the second equation of (15.39) and the assumption that  $\nu$  and  $\tau$  are stopping times, we see that  $A_k \in \mathcal{F}_{k-1}$ . By the boundedness of  $\nu$  and  $\tau$  we can choose a constant  $N$  such that  $\nu \leq N$  and  $\tau \leq N$ , and the definition of the transformed process  $\{\widetilde{M}_n : n \geq 0\}$  then gives us

$$M_\tau - M_\nu = \widetilde{M}_N.$$

By Problem 2.2 the process  $\{\widetilde{M}_n : n \geq 0\}$  is a submartingale, so we have  $0 = E(\widetilde{M}_0) \leq E(\widetilde{M}_N)$  which now entails  $E(M_\nu) \leq E(M_\tau)$ .

SOLUTION FOR PROBLEM 2.4. If we set  $X_i = \mathbb{I}(A_i) - P(A_i)$  and form the sum  $M_n = X_1 + X_2 + \cdots + X_n$ , then  $M_n$  is a martingale. By Doob's stopping time theorem we then have  $E(M_{n \wedge \tau_k}) = 0$ , or, to be explicit,

$$E\left(\sum_{i=1}^{n \wedge \tau_k} P(A_i)\right) = E\left(\sum_{i=1}^{n \wedge \tau_k} \mathbb{I}(A_i)\right) \quad \text{for all } n. \quad (15.40)$$

Now, by the monotone convergence theorem and the finiteness of  $\tau_k$ , we have

$$\lim_{n \rightarrow \infty} E\left(\sum_{i=1}^{n \wedge \tau_k} P(A_i)\right) = E\left(\lim_{n \rightarrow \infty} \sum_{i=1}^{n \wedge \tau_k} P(A_i)\right) = E\left(\sum_{i=1}^{\tau_k} P(A_i)\right) = E[\phi(\tau_k)],$$

while by the monotone convergence theorem and the definition of  $\tau_k$ , we have

$$\lim_{n \rightarrow \infty} E\left(\sum_{i=1}^{n \wedge \tau_k} \mathbb{I}(A_i)\right) = E\left(\lim_{n \rightarrow \infty} \sum_{i=1}^{n \wedge \tau_k} \mathbb{I}(A_i)\right) = E(k) = k,$$

Therefore, by taking limits in (15.40), we obtain our target identity (2.30).

SOLUTION FOR PROBLEM 2.5. Induction and the recursive definition (2.31) of  $A_n$  make the first two properties obvious. We really only need to check that when  $N_n$  is defined by setting  $N_n = M_n^2 - A_n$  the process  $\{N_n\}$  is a martingale. For this, we just note

$$\begin{aligned} E(N_{n+1}|\mathcal{F}_n) &= E(M_{n+1}^2 - A_{n+1}|\mathcal{F}_n) = E(M_{n+1}^2|\mathcal{F}_n) - A_{n+1} \\ &= E(M_{n+1}^2|\mathcal{F}_n) - A_n - E[(M_{n+1} - M_n)^2|\mathcal{F}_n] \\ &= 2E(M_{n+1}M_n|\mathcal{F}_n) - M_n^2 - A_n = M_n^2 - A_n = N_n, \end{aligned}$$

where in the last line we used  $E(M_{n+1}M_n|\mathcal{F}_n) = M_nE(M_{n+1}|\mathcal{F}_n) = M_n^2$ .

SOLUTION FOR PROBLEM 2.6. If  $X$  equals  $x$  with probability  $1/2$  and equals  $y$  with probability  $1/2$ , then Jensen's inequality tells us that  $E(|X|)^p \leq E(|X|^p)$ , so, when we work on the expectations, we have

$$\left(\frac{|x| + |y|}{2}\right)^p \leq \frac{|x|^p + |y|^p}{2},$$

which may be rearranged to give the bound (2.32).

SOLUTION FOR PROBLEM 2.7. For the first part, we take a hint from the proof of Doob's inequality (especially the formula (2.22) page 29); we multiply the hypothesis (2.33) by  $p\lambda^{p-1}$  and integrate. From the hypothesis (2.33) we have

$$p\lambda^{p-1}P(X/3 \geq \lambda) \leq p\lambda^{p-1}P(Y \geq \lambda),$$

so by integration over  $\lambda \in [0, \infty)$  we find  $E[(X/3)^p] \leq E[(Y/7)^p]$ , which is just what we needed.

Next, since  $P(X \geq 2\lambda) \leq P(X \geq 2\lambda, Y \leq \lambda) + P(Y \geq \lambda)$ , the hypothesis of Part (b) gives us

$$4\lambda^3P(X/2 \geq \lambda) \leq \frac{1}{20}4\lambda^3P(X \geq \lambda) + 4\lambda^3P(Y \geq \lambda).$$

This time when we integrate we find

$$E[(X/2)^4] \leq \frac{1}{20}E(X^4) + E(Y^4),$$

which quickly simplifies to give  $E(X^4) \leq 80E(Y^4)$ .

SOLUTION FOR PROBLEM 2.8. Starting with Fatou's Lemma we find

$$\begin{aligned} E(|M_\tau|\mathbb{I}(\tau < \infty)) &= E(\lim_{n \rightarrow \infty} |M_{n \wedge \tau}\mathbb{I}(\tau < \infty)|) \leq \liminf_{n \rightarrow \infty} E(|M_{n \wedge \tau}|) \\ &\leq \sup E(|M_{n \wedge \tau}|) \leq \sup E(|M_n|) \end{aligned}$$

where in the last step we used Exercise 2.3 and the fact that  $|M_{n \wedge \tau}|$  is a submartingale by Doob's stopping time theorem. Remark: Lamb (1973) uses this inequality to give a very brief proof of the convergence theorem for  $L^1$  bounded martingales.

SOLUTION FOR PROBLEM 2.9. We have  $E(|M_n|) \leq E(|M_n|^p)^{1/p} \leq B^{1/p} < \infty$  by Jensen's inequality, so  $\{M_n : n \geq 0\}$  is also an  $L^1$ -bounded martingale. The  $L^1$  convergence theorem tells us that there is an  $M_\infty \in L^1$  such that  $M_n$  converges almost surely to  $M_\infty$ , confirming the first assertion of (2.35).

To prove the second assertion, we first note by Fatou's Lemma that

$$E(|M_\infty|^p) = E(\liminf_{n \rightarrow \infty} |M_n|^p) \leq \liminf_{n \rightarrow \infty} E(|M_n|^p) \leq B < \infty,$$

so  $M_\infty \in L^p$ . If we set  $D = \sup_n |M_n|$ , then  $D \in L^p$  by Doob's maximal inequality (2.23), so by the elementary Jensen bound (2.32), we have

$$|M_n - M_\infty|^p \leq 2^{p-1} \{|M_n|^p + |M_\infty|^p\} \leq 2^{p-1} \{D^p + |M_\infty|^p\}.$$

Since  $D^p + |M_\infty|^p \in L^1$ , we can therefore apply the dominated convergence theorem to the sequence  $|M_n - M_\infty|^p$  to obtain

$$\lim_{n \rightarrow \infty} E(|M_n - M_\infty|^p) = E(\lim_{n \rightarrow \infty} |M_n - M_\infty|^p) = 0;$$

in other words,  $\lim_{n \rightarrow \infty} \|M_n - M_\infty\|_p = 0$ , just as we hoped.

### Chapter 3

SOLUTION FOR PROBLEM 3.1. For part (a), complete the square in the exponent to get

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} e^{-x^2/2} dx = e^{t^2/2} \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-t)^2/2} dx \right\}$$

and observe that the braced integral is equal to one since  $e^{-(x-t)^2/2}/\sqrt{2\pi}$  is the density of a Gaussian random variable with mean  $\mu = t$  and variance 1.

For (b) one can equate the coefficients in

$$E[e^{tX}] = e^{t^2/2} \Leftrightarrow \sum_{n=0}^{\infty} E(X^n) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{t^{2n}}{2^n n!}, \quad (15.41)$$

or one can use integration by parts to get a recursion,

$$\begin{aligned} E(X^{2n}) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^{2n} e^{-x^2/2} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^{2n-1} (x e^{-x^2/2}) dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (2n-1) x^{2n-2} e^{-x^2/2} dx = (2n-1) E(X^{2n-2}). \end{aligned}$$

Finally, for (c), one expands, uses the moments, and recognizes the sum:

$$E(e^{itX}) = \sum_{n=0}^{\infty} E(X^n) \frac{i^n t^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2^n n!} = e^{-t^2/2}.$$

For an unlikely (but instructive) alternative, one can also prove (c) by noting

$$f(t) = E(e^{itX}) = \operatorname{Re} E(e^{itX}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} \cos(tx) dx,$$

so, by differentiation under the integral and integration by parts, one has