Appendix I: Problem Hints and Solutions

Chapter 1

Solution for Problem 1.1. Let $T_{i,j}$ denote the expected time to go from level i to level j, and note by formula (1.16) that $T_{25,20} = 15$ and $T_{21,20} = 3$. check! By first-step analysis we also have

$$T_{20,19} = \frac{1}{10} \cdot 1 + \frac{9}{10} \cdot \{1 + T_{21,20} + T_{20,19}\}$$

so substituting $T_{21,20} = 3$ and solving gives $T_{20,19} = 37$. Similarly, we have

$$T_{19,18} = \frac{1}{3} \cdot 1 + \frac{2}{3} \cdot \{1 + T_{20,19} + T_{19,18}\},$$

so substituting $T_{20,19} = 37$ and solving gives $T_{19,18} = 77$. Finally, one finds

$$T_{25,18} = T_{25,20} + T_{20,19} + T_{19,18} = 15 + 37 + 77 = 129.$$

Solution for Problem 1.2. We get (1.26) just by substitution, and we also have

$$N_n = \sum_{k:2k \le n} \mathbb{I}(S_{2k} = 0) \quad \text{and} \quad E(N_n) = \sum_{k:2k \le n} P(S_{2k} = 0),$$

so (1.26) and integral comparison give us

$$\sum_{k:2k \le n} P(S_{2k} = 0) \sim \sum_{1 \le k \le n/2} 1/\sqrt{\pi k} \sim \sqrt{2n/\pi} \quad \text{as } n \to \infty.$$

Solution for Problem 1.3. First set $N_\infty = \lim N_n$ and then note that $P(N_\infty \ge k) = r^k$ since the event $\{N_\infty \ge k\}$ entails k successes of independent events each of which has success probability r. Now, if it were truely the case that $0 \le r < 1$, then we would have

$$E(N_n) \le E(N_\infty) = \sum_{k=1}^{\infty} P(N_\infty \ge k) = \sum_{k=1}^{\infty} r^k = \frac{r}{1-r} < \infty,$$

but by Exercise 1.2 we know $E(N_n) \sim \sqrt{2n/\pi}$, so we must have $r = 1$.

Solution for Problem 1.4. First-step analysis gives us

$$P(\tau_0 = 2k) = \frac{1}{2} P(\tau_0 = 2k \mid X_1 = 1) + \frac{1}{2} P(\tau_0 = 2k \mid X_1 = -1),$$

but by symmetry and the identity (1.24) we have

$$P(\tau_0 = 2k \mid X_1 = 1) = P(\tau_0 = 2k \mid X_1 = -1) = \frac{1}{2k-1} \binom{2k}{k} 2^{-2k}$$

from which (1.28) follows. The asymptotic formula (1.29) follows directly from Stirling's formula, and the relation $E(\tau_0) = \infty$ is also straightforward. With just a little more work, one can use (1.29) to check that $E(\tau_0^\alpha) < \infty$ for all $\alpha < \frac{1}{2}$ and that $E(\tau_0^\alpha) = \infty$ for all $\alpha \ge \frac{1}{2}$.

Solution for Problem 1.5. To prove the first identity of (1.31), note that for $k \ge 1$ the event $\{L_k > 0\}$ cannot occur unless the first step of the random walk is to $+1$. If the first step is to $+1$, then $\{L_k > 0\}$ occurs if and only if the walk hits k before it hits 0. By (1.2) this occurs with probability $1/k$. When we put these two independent requirements together, we see that $P(L_k > 0) = (1/2)(1/k)$.

To prove the second identity of (1.31), we consider a time at which the walk hits level k, and we make two observations. If on its next step the walk goes up, then it is guaranteed to hit level k at least one more time before it hits level 0. On the other hand, if the walk goes down on the next step after hitting level k, then by (1.2) the walk will hit level k a least one more time with probability $(k-1)/k$. These observations combine to give us (1.31).

Finally, to prove (1.30), we note that

$$\begin{aligned} P(N_k > j) &= P(N_k > 0) P(N_k > 1 \mid N_k > 0) \cdots P(N_k > j \mid N_k > j-1) \\ &= \frac{1}{2k} \left(\frac{1}{2} + \frac{1}{2} \frac{k-1}{k} \right)^j = \frac{1}{2k} \left(\frac{2k-1}{2k} \right)^j . \end{aligned}$$

If we now sum over $0 \le j < \infty$ we get $E(N_k)$ on the left, while on the right we see that geometric summation gives us exactly 1.

Incidentally, this problem has an elegant generalization to biased random walk where $p < q$. If we repeat our argument but use the ruin probability formula (1.13) in place of the formula (1.2) for unbiased ruin probabilities, we discover that $E(L_k) = (p/q)^k$. For $p = q$ this recaptures the formula (1.30), and, in a way, it explains why $E(L_k)$ does not depend on k for unbiased walk.

Solution for Problem 1.6. Let $N_{(1,1)(\alpha+\beta,\alpha-\beta)}$ denote the total number of random walk paths from $(1,1)$ to $(\alpha+\beta, \alpha-\beta)$ and let

$$N^{\text{DoTouch}}_{(1,1)(\alpha+\beta,\alpha-\beta)} \quad \text{and} \quad N^{\text{DoNotTouch}}_{(1,1)(\alpha+\beta,\alpha-\beta)}$$

denote the corresponding number of paths that respectively *do* and *do not* touch the axis. By the reflection principle and path counting we find

$$\begin{aligned} N^{\text{DoNotTouch}}_{(1,1)(\alpha+\beta,\alpha-\beta)} &= N_{(1,1)(\alpha+\beta,\alpha-\beta)} - N^{\text{DoTouch}}_{(1,1)(\alpha+\beta,\alpha-\beta)} \\ &= N_{(1,1)(\alpha+\beta,\alpha-\beta)} - N_{(1,-1)(\alpha+\beta,\alpha-\beta)} \\ &= \binom{\alpha+\beta-1}{\alpha-1} - \binom{\alpha+\beta-1}{\alpha} = \frac{\alpha-\beta}{\alpha+\beta}\binom{\alpha+\beta}{\alpha}, \end{aligned}$$

so the probability that A leads throughout the counting process is

$$N^{\text{DoNotTouch}}_{(1,1)(\alpha+\beta,\alpha-\beta)}/N_{(0,0)(\alpha+\beta,\alpha-\beta)} = N^{\text{DoNotTouch}}_{(1,1)(\alpha+\beta,\alpha-\beta)}/\binom{\alpha+\beta}{\alpha} = \frac{\alpha-\beta}{\alpha+\beta}.$$

Chapter 2

Solution for Problem 2.1. The solutions of the equation

$$1 = E(y^{X_1}) = 0.52\, y^{-1} + 0.45y + 0.03y^2$$

are $y = 1$, $y = 1.01849$, and $y = -17.01849$, and for any one of these $M_n = y^{S_n}$ is a martingale. Since $y = 1$ gives a trivial martingale and since $y = -17.01849$ becomes unruly when raised to a high power, we take $y = 1.01849$ to define M_n. We then argue as before that $E(M_\tau) = 1$, and this gives us

$$1 = y^{100} P(S_\tau = 100) + y^{101} P(S_\tau = 101) + y^{-100} P(S_\tau = -100).$$

Now, if we let $p = P(S_\tau = 100) + P(S_\tau = 101)$, the fact that $y > 1$ gives us

$$1 < y^{101}\, p + y^{-100}(1-p) \quad \text{and} \quad 1 > y^{100} p + y^{-100}(1-p).$$

Solving for p and substituting for y gives

$$\frac{1-y^{-100}}{y^{101}-y^{-100}} < p < \frac{1-y^{-100}}{y^{100}-y^{-100}} \quad \text{or} \quad 0.1353 < p < 0.1379,$$

so p is determined within an error of 3×10^{-3}.

Incidentally, by taking advantage of high-precision arithmetic (say as provided by *Mathematica*), one can use the *pair* of martingales determined by $y = 1.01849$ and $y = -17.01849$ to obtain a system of two equations in two unknowns which can be solved exactly for all of the values $P(S_\tau = 100)$, $P(S_\tau = 101)$, and $P(S_\tau = -100)$. This pleasing trick often helps, and it suggests a general principle: Two martingales can be better than one! The full answer?

Solution for Problem 2.2. Since $\{A_n\}$ is bounded and adapted, we see that $\widetilde{M}_n$ is in $\mathcal{F}_n$ and integrable for each $n \geq 0$. To check the martingale identity we note

$$
\begin{aligned}
E(\widetilde{M}_n \mid \mathcal{F}_{n-1}) &= E\big(\widetilde{M}_{n-1} + A_n(M_n - M_{n-1}) \mid \mathcal{F}_{n-1}\big) \\
&= \widetilde{M}_{n-1} + A_n E\big(M_n - M_{n-1} \mid \mathcal{F}_{n-1}\big) \geq \widetilde{M}_{n-1}
\end{aligned}
$$

since $A_n E\big(M_n - M_{n-1} \mid \mathcal{F}_{n-1}\big)$ is nonnegative.

Solution for Problem 2.3. First we notice that we can assume without loss of generality that $M_0 = 0$ since

$$E[M_\nu] \leq E[M_\tau] \quad \Longleftrightarrow \quad E[M_\nu - M_0] \leq E[M_\tau - M_0].$$

Now we apply the result of Problem 2.2 with the choice

$$A_k = \mathbb{I}(\nu < k \leq \tau) = 1 - \mathbb{I}(\tau \leq k-1) - \mathbb{I}(\nu \leq k-1). \tag{15.39}$$

By the first equation of (15.39), we see that A_k is nonnegative and bounded. By the second equation of (15.39) and the assumption that ν and τ are stopping times, we see that $A_k \in \mathcal{F}_{k-1}$. By the boundedness of ν and τ we can choose a constant N such that $\nu \leq N$ and $\tau \leq N$, and the definition of the transformed process $\{\widetilde{M}_n : n \geq 0\}$ then gives us

$$M_\tau - M_\nu = \widetilde{M}_N.$$

By Problem 2.2 the process $\{\widetilde{M}_n : n \geq 0\}$ is a submartingale, so we have $0 = E(\widetilde{M}_0) \leq E(\widetilde{M}_N)$ which now entails $E(M_\nu) \leq E(M_\tau)$.

Solution for Problem 2.4. If we set $X_i = \mathbb{I}(A_i) - P(A_i)$ and form the sum $M_n = X_1 + X_2 + \cdots + X_n$, then M_n is a martingale. By Doob's stopping time theorem we then have $E(M_{n\wedge\tau_k}) = 0$, or, to be explicit,

$$E\bigg(\sum_{i=1}^{n\wedge\tau_k} P(A_i)\bigg) = E\bigg(\sum_{i=1}^{n\wedge\tau_k} \mathbb{I}(A_i)\bigg) \qquad \text{for all } n. \tag{15.40}$$

Now, by the monotone convergence theorem and the finiteness of τ_k, we have

$$\lim_{n\to\infty} E\bigg(\sum_{i=1}^{n\wedge\tau_k} P(A_i)\bigg) = E\bigg(\lim_{n\to\infty} \sum_{i=1}^{n\wedge\tau_k} P(A_i)\bigg) = E\bigg(\sum_{i=1}^{\tau_k} P(A_i)\bigg) = E[\phi(\tau_k)],$$

while by the monotone convergence theorem and the definition of τ_k, we have

$$\lim_{n\to\infty} E\bigg(\sum_{i=1}^{n\wedge\tau_k} \mathbb{I}(A_i)\bigg) = E\bigg(\lim_{n\to\infty} \sum_{i=1}^{n\wedge\tau_k} \mathbb{I}(A_i)\bigg) = E(k) = k,$$

Therefore, by taking limits in (15.40), we obtain our target identity (2.30).

Solution for Problem 2.5. Induction and the recursive definition (2.31) of A_n make the first two properties obvious. We really only need to check that when N_n is defined by setting $N_n = M_n^2 - A_n$ the process $\{N_n\}$ is a martingale. For this, we just note

$$
\begin{aligned}
E(N_{n+1}|\mathcal{F}_n) &= E(M_{n+1}^2 - A_{n+1}|\mathcal{F}_n) = E(M_{n+1}^2|\mathcal{F}_n) - A_{n+1} \\
&= E(M_{n+1}^2|\mathcal{F}_n) - A_n - E[(M_{n+1} - M_n)^2|\mathcal{F}_n] \\
&= 2E(M_{n+1}M_n|\mathcal{F}_n) - M_n^2 - A_n = M_n^2 - A_n = N_n,
\end{aligned}
$$

where in the last line we used $E(M_{n+1}M_n|\mathcal{F}_n) = M_n E(M_{n+1}|\mathcal{F}_n) = M_n^2$.

Solution for Problem 2.6. If X equals x with probability $1/2$ and equals y with probability $1/2$, then Jensen's inequality tells us that $E(|X|)^p \le E(|X|^p)$, so, when we work or the expectations, we have

$$
\left(\frac{|x| + |y|}{2}\right)^p \le \frac{|x|^p + |y|^p}{2},
$$

which may be rearranged to give the bound (2.32).

Solution for Problem 2.7. For the first part, we take a hint from the proof of Doob's inequality (especially the formula (2.22) page 29); we multiply the hypothesis (2.33) by $p\lambda^{p-1}$ and integrate. From the hypothesis (2.33) we have

$$
p\lambda^{p-1}P(X/3 \ge \lambda) \le p\lambda^{p-1}P(Y/ \ge \lambda),
$$

so by integration over $\lambda \in [0, \infty)$ we find $E\big[(X/3)^p\big] \le E\big[(Y/7)^p\big]$, which is just what we needed.

Next, since $P(X \ge 2\lambda) \le P(X \ge 2\lambda, Y \le \lambda) + P(Y \ge \lambda)$, the hypothesis of Part (b) gives us

$$
4\lambda^3 P(X/2 \ge \lambda) \le \frac{1}{20} 4\lambda^3 P(X \ge \lambda) + 4\lambda^3 P(Y \ge \lambda).
$$

This time when we integrate we find

$$
E\big[(X/2)^4\big] \le \frac{1}{20} E(X^4) + E(Y^4),
$$

which quickly simplifies to give $E(X^4) \le 80E(Y^4)$.

Solution for Problem 2.8. Starting with Fatou's Lemma we find

$$
\begin{aligned}
E\left(|M_\tau|\mathbb{I}(\tau < \infty)\right) &= E(\lim_{n\to\infty} |M_{n\wedge\tau}\mathbb{I}(\tau < \infty)|) \le \liminf_{n\to\infty} E(|M_{n\wedge\tau}|) \\
&\le \sup E(|M_{n\wedge\tau}|) \le \sup E(|M_n|)
\end{aligned}
$$

where in the last step we used Exercise 2.3 and the fact that $|M_{n\wedge\tau}|$ is a submartingale by Doob's stopping time theorem. Remark: Lamb (1973) uses this inequality to give a very brief proof of the convergence theorem for L^1 bounded martingales.

Solution for Problem 2.9. We have $E(|M_n|) \le E(|M_n|^p)^{1/p} \le B^{1/p} < \infty$ by Jensen's inequality, so $\{M_n : n \ge 0\}$ is also an L^1-bounded martingale. The L^1 convergence theorem tells us that there is an $M_\infty \in L^1$ such that M_n converges almost surely to M_∞, confirming the first assertion of (2.35).

To prove the second assertion, we first note by Fatou's Lemma that

$$E(|M_\infty|^p) = E(\liminf_{n\to\infty} |M_n|^p) \le \liminf_{n\to\infty} E(|M_n|^p) \le B < \infty,$$

so $M_\infty \in L^p$. If we set $D = \sup_n |M_n|$, then $D \in L^p$ by Doob's maximal inequality (2.23), so by the elementary Jensen bound (2.32), we have

$$|M_n - M_\infty|^p \le 2^{p-1}\{|M_n|^p + |M_\infty|^p\} \le 2^{p-1}\{D^p + |M_\infty|^p\}.$$

Since $D^p + |M_\infty|^p \in L^1$, we can therefore apply the dominated convergence theorem to the sequence $|M_n - M_\infty|^p$ to obtain

$$\lim_{n\to\infty} E(|M_n - M_\infty|^p) = E(\lim_{n\to\infty} |M_n - M_\infty|^p) = 0;$$

in other words, $\lim_{n\to\infty} ||M_n - M_\infty||_p = 0$, just as we hoped.

Chapter 3

Solution for Problem 3.1. For part (a), complete the square in the exponent to get

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} e^{-x^2/2}\, dx = e^{t^2/2} \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-t)^2/2}\, dx \right\}$$

and observe that the braced integral is equal to one since $e^{-(x-t)^2/2}/\sqrt{2\pi}$ is the density of a Gaussian random variable with mean $\mu = t$ and variance 1.

For (b) one can equate the coefficients in

$$E[e^{tX}] = e^{t^2/2} \quad \Leftrightarrow \quad \sum_{n=0}^{\infty} E(X^n) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{t^{2n}}{2^n n!}, \tag{15.41}$$

or one can use integration by parts to get a recursion,

$$\begin{aligned} E(X^{2n}) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^{2n} e^{-x^2/2}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^{2n-1} (x e^{-x^2/2})\, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (2n-1) x^{2n-2} e^{-x^2/2}\, dx = (2n-1) E(X^{2n-2}). \end{aligned}$$

Finally, for (c), one expands, uses the moments, and recognizes the sum:

$$E(e^{itX}) = \sum_{n=0}^{\infty} E(X^n) \frac{i^n t^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2^n n!} = e^{-t^2/2}.$$

For an unlikely (but instructive) alternative, one can also prove (c) by noting

$$f(t) = E(e^{itX}) = \operatorname{Re} E(e^{itX}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} \cos(tx)\, dx,$$

so, by differentiation under the integral and integration by parts, one has