Stochastic Calculus and Financial Applications
Final Take Home Exam (Steele: Fall 2006)

INSTRUCTIONS. You may consult any books or articles that you find useful. If you use a result that is not from our text, attach a copy of the relevant pages from your source. You may use any software, including the internet, Mathematica, Maple, R, S-Plus, MatLab, etc. **Attach any Mathematica (or similar) code that you use.**

- You may **NOT** consult with any other person about these problems. If you have a question, even one that is just about the meaning of a question, please contact me directly rather than consult with a fellow student.
- I may post "bug reports" or clarifications on our web page, and you should regularly check for these.
- You should strive to make your answers as **clear and complete** as possible. Neatness counts — especially of thought, but even of handwriting. If I can't read it, I can't grade it.
- Never, ever, write down anything that you know — or even vaguely suspect — to be false. If you understand that your argument is incomplete or only heuristic, this may be fine, but it should be properly labeled as incomplete or heuristic.
- Don't skip steps. If I can't go from line n to line $n+1$ in my head, something is missing. If you use Mathematica or a fact from a table, please say so **and document it.** Otherwise, I stare and stare at line n wondering how you got to $n+1$ in your head while I can't.
- Use anything from anyplace, but do not steal. If you make use of an argument from some source, give credit to the source. If you find the complete (and correct!) solution to a problem in a book or on the internet, just print out the pages and attach them. You will get full credit.
- Write on only one side of a page. Use decent, homogeneous, high quality paper. No dinner napkins, hoagie sacks, *Indian Chief* Tablets, etc.
- Begin each new solution on a new page.
- Arrange your solutions in the natural numerical order. If you do not do problem K, then include a self-standing page that says "Problem K was not done."
- Staple your pages neatly with a high-quality stapler with appropriate length and weight to do a clean and secure job.
- **As discussed in class, you MUST use and complete the cover page given at the website. Self-evaluation is hugely valuable.**

GENERAL ADVICE

1. In your solutions, please do not just write down things that you think are relevant even though they do not add up to an honest solution. Such lists are useful when you are working on a problem, but if you offer list as a solution your keeping yourself from having the "missing" idea.

2. If you can explain **clearly** something that you tried that did not work, this sometimes is worth a few points. Please do not abuse this offer. With experience, one learns that many sensible ideas do not work. Almost by definition, this is what separates the trivial from the non-trivial.

3. Try to keep in mind that a good problem requires that one "overcome some objection." What distinguishes a problem from an exercise is that in a good problem a routine plan does not work. The whole point is to go past the place where routine ideas take you. Still, don't shy away from the obvious; **almost all of the "problems" here are "exercises."**

4. If you do something **extra** that is valid, you can get "bonus" points. These special rewards cannot be determined in advance. They are usual small, but they can be substantial — and they do add up.

5. The most common source of bonus points is for saying something particularly well. Clear, well-organize, solutions are gems. They deserve to be acknowledged.

THE BIG PICTURE

Almost certainly these instructions will seem to be overly detailed to you. It is true that they are detailed, but they evolved case by case. Each rule deals with some previous misunderstanding. When you start teaching (and grading) I encourage you to follow this example. There is no harm in a having a few creative (yet compassionate) eccentricities.

There is a final motivation for this long list of rules and suggestions. **Detailed instructions provide clear coaching for excellence.** We all do wonderfully better when we are lucky enough to know what we need to do. This is the kind of break one seldom gets in research.

Due Date and Place: The exam **with its completed self-evaluation cover sheet** is due in my office JMHH 447 at 11am on Tuesday December 19. Solutions may be emailed or sent by FedEx. Fax is not acceptable. The preferred delivery method is a hard copy in my hand.

Problems for Everyone

Problem 1: A Construction of Brownian Motion on $[0, \infty)$

Given that a Brownian motion $\{B(t) : 0 \leq t \leq 1\}$ has been defined on the unit interval, define an new process on all of $0 \leq t < \infty$ by setting

$$\tilde{B}(t) \equiv (1+t)\left\{ B\left(\frac{t}{1+t}\right) - \frac{tB(1)}{1+t} \right\}.$$

Confirm that this process is indeed a Brownian motion on $[0, \infty)$. Although this construction of Brownian motion on $[0, \infty)$ has fewer seams than the one given in the text, it is probably less obvious to most people.

Problem 2: Intuition about Volatility from Three Perspectives

Consider a European call option with the current stock price equal to the current strike price. These are commonly called at-the-money options, though there are more sophisticated definitions of "at-the-money."

(a) Now suppose that volatility is zero. If you make the usual Black-Scholes assumptions, explain how one can guess the value of the option by direct reasoning without using either the Black-Scholes formula or the Black-Scholes PDE.

(b) Now take the Black-Scholes PDE and let $\sigma = 0$. Does the value that you obtained in Part (a) solve that simplified PDE and its terminal condition?

(c) Finally, take the Black-Scholes formula, and calculate what you get when $\sigma \to 0$. Be sure to handle any indeterminate expressions honestly.

Incidentally, Richard Feynman once said that you understand an equation when "without solving the equation you can still say how the solutions behave." Feynman's criterion is worth keeping in mind anytime one meets an equation — new or old.

Problem 3: Is it Brownian Motion?

(a) Consider the process, $X_t = B_{2t} - B_t$, $0 \leq t < \infty$. Is it a Gaussian process? Can you find the mean and variance? Is it Brownian motion?

(b) Let X_t and Y_t be independent Brownian motions. Let $Z_t = (X_t + Y_t)/\sqrt{2}$. Is it a Gaussian process? Can you find the mean and variance? Is it Brownian motion?

Problem 4: Integrand and Integral Size

Suppose that $\phi(\omega, t)$ is a stochastic integrand for which there is a constant C such that

$$E(\phi^2(\omega, t)) \leq Ct^p \quad \text{for all } t \geq 0. \tag{1}$$

Show that the stochastic integral

$$X_t = \int_0^t \phi(\omega, s)\, dB_s$$

satisfies the bound

$$E(|X_t|) \le \left(\frac{C}{p+1}\right)^{1/2} t^{(p+1)/2}.$$

Problem 5: A Greek Bound on the Call Price

Let $f(t, S_t)$ denotes the time t arbitrage-free price of a call option under the usual Black–Scholes model, and explain why one has the bound

$$f(t, S_t) \le S_t \Delta_t + \frac{\sigma^2}{2r} S_t^2 \Gamma_t,$$

where, as usual, we have $\Delta_t = f_x(t, S_t)$ and $\Gamma_t = f_{xx}(t, S_t)$.

Problem 6: A Text Book Blooper

A recently published text asserts that for each fixed $\gamma > 0$ the collection of random variables

$$\mathcal{C} = \{X_t = \exp(\gamma B_t - \gamma^2 t/2) \, : 0 \le t < \infty\}$$

is uniformly integrable. Prove that this statement is false.

Curiously enough, $\mathcal{C}$ is almost a "poster child" for a collection that fails to be uniformly integrable. Dozens of papers offer conditions on a stopping time τ that suffice to give $E[\exp(B_\tau - \tau/2)] = 1$, and all of these would be irrelevant if $\mathcal{C}$ were a uniformly integrable collection.

Problem 7: A Hitting Time Identity

Let $\tau = \min\{t : B_t = 1\}$, and show that we have the nice identity

$$E[\,\exp(-\tau/2)\,] = \frac{1}{e}.$$

We know the density of τ, but direct integration would not be the most pleasing way to obtain this expectation. What's wanted here is a simple martingale argument. Incidentally, this formula adds something to our intuition about τ. We know that $E(\tau) = \infty$, but this new formula says that "properly discounted" τ is "around" 2.

Problem 8: A Martingale and an Integral

Show that if g is a continuously differentiable function, then the process

$$M_t = g(t)B_t - \int_0^t g'(s) B_s \, ds$$

is a martingale. Next, show that for $\tau = \min\{t : B_t = A \text{ or } B_t = -B\}$ one has

$$E[\,\sin(\tau)B_\tau] = E\left(\int_0^\tau \cos(s) B_s \, ds\right). \qquad (2)$$

Problem 9: All that Glitters is not Brownian Motion

Suppose that X_t and Y_t are two continuous processes such that for each pair of constants α and β with $\alpha^2+\beta^2=1$ the process $Z_t=\alpha X_t+\beta Y_t$ is a standard Brownian motion. First, show that for each t the random variables X_t and Y_t are independent. Second, show by example that the processes $\{X_t\}$ and $\{Y_t\}$ need not be independent.

Big Hint: OK, for part two, I'll show you the example, but you still have to check that it works. Consider independent Brownian motions $B_1(t)$ and $B_2(t)$, and then set $X_t = B_1(2t/3) - B_2(t/3)$ and $Y_t = B_1(t/3) + B_2(2t/3)$.

Problems with a Challenge, but Still Solidly in Range

Problem 10: The Black–Scholes "Lower Case" Transformation

Take the Black–Scholes formula, replace Φ by ϕ, and simplify. I mean *really simplify*!. What do you get? I promise that you will remember the stunning answer for the rest of your life. It's also financially informative. Show your work!

Problem 11: Texas Chain Rule Massacre

Suppose that $f(t,x)$ solves the Black-Scholes PDE with the usual terminal condition $f(T,x)=(x-K)_+$. Consider the new variable y and the new function $g(t,y)$ that are defined by the relations:

$$y = e^{r(T-t)}x \quad \text{and} \quad g(t,y) = e^{r(T-t)}f(t,x).$$

Show that g satisfies the more pleasant terminal value PDE

$$\frac{\partial g}{\partial t} + \frac{1}{2}\sigma^2 y^2 \frac{\partial^2 g}{\partial^2 y} = 0 \quad \text{and} \quad g(T,y) = (y-K)_+. \tag{3}$$

This certainly a nice simplification of the Black-Scholes equation, and it suggests that the new variables are "better" variables. With time on our hands, we could see what other simplifications might be achieved with using y and g in place of x and f.

Hint: This is only a chain rule exercise, but it is easily messed up. If you make a mistake in your calculations and still arrive at the target formula, you may have engaged in a willful fiction — which is not polite.

To succeed here is easy, if you start out right. The right way to start is to first write the "old stuff" as a function of the "new stuff." You can then do straightforward chain rule calculations of the derivatives associated with the "old stuff," plug these into the "old equation," and get the "new equation." Less systematic approaches are often fought both with error and irrelevant algebra.

Problem 12: One of Two Lines

Consider the standard Brownian motion $Z_t = (X_t, Y_t)$ in $\mathbb{R}^2$, and let τ denote the first time that either $Y_t = aX_t + b$ or $Y_t = aX_t - b$. That is, we let τ denote the first time that Z_t hits the boundary of the strip S defined by the parallel lines $y = ax + b$ and $y = ax - b$. Show that one has

$$E(\tau) = b^2/(1 + a^2),$$

then pose — and prove — a three dimensional generalization.

Problem 13: BM without Assuming Independence

Let $\{X_t\}$ denote a mean zero Gaussian process and let $\Delta(s, t) = X_t - X_s$ denote the increment in the process from time s to time t. Assume that for each choice of four times $0 \leq t_1 \leq t_2 \leq t_3 \leq t_4$ the variance of the sum $S = \Delta(t_1, t_2) + \Delta(t_2, t_3) + \Delta(t_3, t_4)$ is equal to $t_4 - t_1$. Show that $\{X_t\}$ is Brownian motion. The point here is that our assumption on S is enough to show that $\{X_t\}$ has independent increments.

Problem 14: A Stopping Time Inequality

Prove that for any stopping time τ and any $t \geq 0$ one has the inequality

$$E\left(\exp(\frac{1}{2}B_{t\wedge\tau})\right) \leq \{E(\exp(\tau/2))\}^{1/2}. \tag{4}$$

Hint: You'll want to get one of your favorite martingales into the game, and, when a square root is in sight, it never hurts to consider Cauchy-Schwarz.

Problem 15: BM and a Stochastic Integral

Consider the process

$$X_t = \int_0^t \operatorname{sign}(B_s)\, dB_s$$

where $\operatorname{sign}(x) = 1$ for $x \geq 0$ and $\operatorname{sign}(x) = -1$ for $x < 0$. Use Levy's theorem to check that X_t is again a Brownian motion. Next, confirm that the two processes X_t and B_t are uncorrelated. Finally show that

$$E(X_t B_t^2) = 2^{5/2} t^{3/2} / 3\sqrt{\pi}, \tag{5}$$

and explain why this implies that the processes X_t and B_t are not independent — despite being uncorrelated and Gaussian processes.

Problem 16: BM and a Stopping Time

Show that for any bounded stopping time τ for the Brownian filtration, one has the bound

$$E(\tau^2) \leq cE(B_\tau^4) \quad \text{where} \quad c = (5 + 2\sqrt{6})/3. \tag{6}$$

Moreover, give an example that shows that this bound can fail for an unbounded stopping time.

Hint: In your investigation you may want to exploit the observation that for any x and y and any $\lambda > 0$ one has $2xy \leq \lambda x^2 + y^2/\lambda$. This is a common device when following a plan where you first estimate with a free parameter and then optimize over the parameter.

Problems that Are a Little Harder

Problem 17: Twin Bounds on an Exit Time

Consider a process X_t with $X_0 = 0$ that satisfies $dX_t = \sigma(X_t)\,dB_t$ where $\sigma(x)$ is a smooth function constrained by the constant bounds

$$0 < \sigma_{\text{low}} \leq \sigma(x) \leq \sigma_{\text{high}} < \infty.$$

Let $A > 0$ and $B > 0$ and set $\tau = \min\{t : X_t = A \text{ or } X_t = -B\}$. After first showing that $E(\tau) < \infty$, show more precisely that one has the twin bounds:

$$AB/\sigma_{\text{high}}^2 \leq E(\tau) \leq AB/\sigma_{\text{low}}^2. \tag{7}$$

Here, one should note that if $\sigma_{\text{high}} = \sigma_{\text{low}}$ then these bounds recover an equality that we have met several times before.

Problem 18: The Famous Ft. Worth Two Step Option

Price an option in Black-Scholes world that pays you one buck at time T if $S_T \geq K_1$ and pays you two bucks if $S_T \geq K_2$ where $0 \leq K_1 < K_2$ are two given "strike prices." Include the details for your computations, or, if you use Mathematica, provide your code and out-put (all cleaned up, but executable). Actually, this computation is easily done by hand and the answer is not more complicated than the Black-Scholes formula.

As a hint, let me suggest that you use the risk-neutral pricing approach rather than the PDE approach. This should make things pretty straightforward. To get the full experience, be sure to include the details of any integrals or probabilities that you compute.

Envoi

I hope that at least some of these problems are interesting to you. Perhaps one or two may even offer a mild epiphany. They have been created for your enjoyment.

These problems should be fully accessible to everyone. Still, easy or hard, I hope that at least a few will scratch out some higher, more conceptual messages. That is the intention behind their design.

There are a few problems that will provide some challenge to almost anyone, but even if you took a brief "break" from the course, don't count yourself out. You can still do all of these problems if that is your desire and if you have time to give them an honestly try.

Still, life is short. You should do the problems you want to do and skip the rest. Whatever problems you chose to solve, I promise to read carefully what you have written carefully. I will do my best to understand your ideas.

Good luck to all!