

Stochastic Calculus and Financial Applications Mid-Term Take Home Exam (Fall 2008)

INSTRUCTIONS. You may consult any books or articles that you find useful. Give complete references for any results that you use. If a result you use is not from our text, attach a copy of the relevant pages from your source. You may use any software, including the internet, Mathematica, etc. Attach any Mathematica (or similar) code that you use.

You may **NOT** consult with any other person about these problems. If you have a question, even one that is just about the meaning of the question, please contact me directly rather than consult with a fellow student. I may post “bug reports” or clarifications on our web page, and you should regularly check for these. Note: The problems are in no particular order; the “easy ones” and the “hard ones” can be any place.

You should strive to make your answers as **clear and complete** as possible. Neatness counts — especially of thought, but even of handwriting. Please write on only one side of a page and please begin each new solution on a new page. Please arrange your solutions in the natural numerical order. If you do not do problem K , then include a self-standing page that says “Problem K was not done.” Staple your pages neatly with a high-quality stapler with appropriate length and weight to do a clean and secure job. As discussed in class, you **MUST** use and complete the cover page given at the website.

These instructions may seem overly detailed, but they evolved like case law. Any “rule” mentioned here is present because of some past mess. When you start teaching (and grading) I encourage you to follow this example. It will save many hours of your life.

Due Date and Place: The exam with its self-evaluation cover sheet is due in class at the beginning of class on Wednesday October 29. I will hand out solutions to the problems at the beginning of the class, and we will spend the class going over them. Naturally, with this structure, late problems would not make much sense.

PROBLEM 1. Consider the process $X_t = \mu t + \sigma B_t$ where μ and σ are constants. First, let A and B be positive constants and set

$$\tau = \min\{t : X_t = A \text{ or } X_t = -B\}.$$

Give a clear, complete, rigorous proof that $P(\tau < \infty) = 1$.

Next, provide a generalization of this result that assumes

$$dX_t = \mu_t(\omega) dt + \sigma_t(\omega) dB_t$$

where $\mu_t(\omega) > 1$ and $\sigma_t(\omega) > 1$ are progressively measurable processes that are suitably integrable.

Finally, take $\mu_t(\omega) \equiv 0$ and consider the condition

$$\int_0^\infty \sigma_t^2(\omega) dt = \infty \quad \text{with probability one.}$$

Is this enough to guarantee that $P(\tau < \infty) = 1$? Provide a proof one way or the other.

PROBLEM 2. Let $\{B_t\}$ be Brownian motion in \mathbb{R}^d , $d \geq 1$, and let

$$\tau_R = \min\{t : |B_t| = R\}.$$

Show that

$$\lim_{R \rightarrow \infty} \tau_R = \infty \quad \text{with probability one.}$$

Can you generalize this result to processes other than Brownian motion? What would be a simple, natural condition on a process $\{X_t\}$ that would suffice here?

PROBLEM 3: A TEXT BOOK BLOOPER

A recently published text asserts that for each fixed $\gamma > 0$ the collection of random variables

$$\mathcal{C} = \{X_t = \exp(\gamma B_t - \gamma^2 t/2) : 0 \leq t < \infty\}$$

is uniformly integrable. Prove that this statement is false.

Curiously enough, \mathcal{C} is almost a “poster child” for a collection that fails to be uniformly integrable. Dozens of papers offer conditions on a stopping time τ that suffice to give $E[\exp(B_\tau - \tau/2)] = 1$, and all of these would be irrelevant if \mathcal{C} were a uniformly integrable collection.

PROBLEM 4. Let $X_t = \mu t + \sigma B_t$ with $\mu > 0$ and $\sigma > 0$. Take $A > 0$ and let $\tau = \min\{t : X_t = A\}$. Find $E(\tau)$ and give a careful derivation of your formula. In particular, justify any interchange of limits.

PROBLEM 5. Let $\tau = \min\{t : B_t = A \text{ or } B_t = -B\}$ and calculate

$$E\left(\int_0^\tau B_t dt\right).$$

PROBLEM 6. Consider the processes defined by the integral

$$X_t = \sqrt{2} t^2 \int_0^\infty B_u e^{-ut} du$$

and the stochastic integral

$$Y_t = \sqrt{2} t \int_0^\infty e^{-ut} dB_u.$$

- (a) Calculate $E(X_t)$, $E(X_t^2)$, $E(Y_t)$, and $E(Y_t^2)$.
- (b) Show that the processes $\{X_t\}$ and $\{Y_t\}$ are actually equivalent. That is, show that for any $0 \leq t_1 < t_2 < \dots < t_n$ the vector $(X_{t_1}, X_{t_2}, \dots, X_{t_n})$ has the same distribution as the vector $(Y_{t_1}, Y_{t_2}, \dots, Y_{t_n})$.
- (c) Show that the processes $Z_t = X_{at}$ and $Z'_t = \sqrt{a}X_t$ are equivalent. Do you know any other process with this scaling property?
- (d) Show that the process defined by $W_0 = 0$ and $W_t = tX_{1/t}$ for $t > 0$ is equivalent to the process $\{X_t\}$. Do you know any other process with this scaling property?

PROBLEM 7. Let $X_t = \sigma B_t$, so X_t is Brownian motion with a scaling factor. Suppose A , and B are nonnegative constants and let

$$\tau = \min\{t : X_t = A \text{ or } X_t = -B\}.$$

- (a) Find $E(\tau^2 X_\tau)$.
- (b) Explain why your answer exhibits appropriate dependence on σ . Also, explain how the dependence on A and B is appropriate.