Stochastic Calculus and Financial Applications
Mid-Term Take Home Exam (Fall 2009)

INSTRUCTIONS. You may consult any books or articles that you find useful. Give complete references for any results that you use. If a result you use is not from our text, attach a copy of the relevant pages from your source. You may use any software, including the internet, Mathematica, etc. Attach any Mathematica (or similar) code that you use.

You may **NOT** consult with any other person about these problems. If you have a question, even one that is just about the meaning of the question, please contact me directly rather than consult with a fellow student. I may post "bug reports" or clarifications on our web page, and you should regularly check for these. Note: The problems are in no particular order; the "easy ones" and the "hard ones" can be any place.

You should strive to make your answers as **clear and complete** as possible. Neatness counts — especially of thought, but even of handwriting. Please write on only one side of a page and please begin each new solution on a new page. Please arrange your solutions in the natural numerical order. If you do not do problem K, then include a self-standing page that says "Problem K was not done." Staple your pages neatly with a high-quality stapler with appropriate length and weight to do a clean and secure job. As discussed in class, you **MUST** use and complete the cover page given at the website.

These instructions may seem overly detailed, but they evolved like case law. Any "rule" mentioned here is present because of some past mess. When you start teaching (and grading) I encourage you to follow this example. It will save many hours of your life.

Due Date and Place: The exam with its self-evaluation cover sheet is due in class at the beginning of class on Monday November 2.

PROBLEM 1. A CONVERSE TO DOOB'S STOPPING TIME THEOREM

Consider a stochastic process $\{M_t : 0 \le t < \infty\}$ that is adapted to the filtration sigma-fields $\{\mathcal{F}_t\}$ and assume that $E(|M_t|) < \infty$ for each $0 \le t < \infty$. Show that if $E(X_\tau) = 0$ for each bounded stopping time τ, then $\{M_t, \mathcal{F}_t\}$ is a martingale.

PROBLEM 2. CONTINUITY OF CONDITIONAL EXPECTATIONS

Consider a filtration $\{\mathcal{F}_t,\, 0 \le t \le 1\}$ that is continuous in the sense that

$$\sigma\{\cup_{s:s<t}\mathcal{F}_s\} = \mathcal{F}_t \quad \text{for each } t \in [0,1].$$

Let X be an integrable random variable, and first show that for each fixed t one has

$$\lim_{s \to t} E(X \,|\, \mathcal{F}_s) = E(X \,|\, \mathcal{F}_t).$$

Next, show that there is a continuous process $\{X_t : 0 \le t \le 1\}$, such that for each $t \in [0,1]$ we have

$$X_t = E(X \,|\, \mathcal{F}_t).$$

PROBLEM 3. SOME CALCULATIONS

(a) Find the distribution of $E(B_t^* \,|\, \mathcal{F}_s)$ for $0 \le s \le t$).

(b) Find the distribution of $E(|B_t| \,|\, \mathcal{F}_s)$ for $0 \le s \le t$).

PROBLEM 4. A LIMIT THEOREM

Consider X_i, normally distributed mean zero variance one, and let S_k denote the sum of the first k of these. Show that the random variables

$$A_n = \frac{1}{n^{3/2}} \sum_{k=0}^{n-1} S_k$$

converge to the distribution of

$$\int_0^1 B_s(\omega)\, ds.$$

PROBLEM 5. ANOTHER DISTRIBUTION Prove or disprove that

$$P(B_t^* - B_t \le x) = P(B_t^* \le x).$$

PROBLEM 6. HITTING TIMES

(a) Let $\tau = \min\{t : |B_t| = a\}$. We know $E(\tau) = a^2$. Find $E(\tau^2)$.

(b) How does $E(\tau^k)$ depend on a? Prove your assertion and try to do so with as little brute force calculation as you can.

Problem 7. Local Martingale or Honest Martingale

Either prove or provide a counter example to the following assertion:

If $\{M_t, \mathcal{F}_t\}$ is a local martingale such that

$$E(M_t^2) < 1 \text{ for all } 0 \leq t \leq T$$

then $\{M_t, \mathcal{F}_t\}$ is a martingale.

Note: If you use a result from a book to help you with this, please provide a full reference and supply any missing calculations. Also, if you find some very original approach (here or in any of the problems) you can get extra credit.