Solution for Problem 2.10. To calculate $E(M_{n+1}|\mathcal{F}_n)$, first note that M_{n+1} is equal to $(A_n+1)/(A_n+B_n+1)$ with probability $M_n = A_n/(A_n+B_n)$ and M_{n+1} equals $A_n/(A_n+B_n+1)$ with probability $1-M_n$. Thus,

$$\begin{aligned} E(M_{n+1}|\mathcal{F}_n) &= \frac{A_n+1}{(A_n+B_n+1)}\frac{A_n}{A_n+B_n} + \frac{A_n}{(A_n+B_n+1)}\frac{B_n}{A_n+B_n} \\ &= \frac{1}{(A_n+B_n+1)}\left\{\frac{A_n(A_n+B_n+1)}{A_n+B_n}\right\} \\ &= \frac{A_n}{A_n+B_n} = M_n. \end{aligned}$$

Since $\{M_n\}$ is bounded between 0 and 1, the martingale convergence theorem tells us that with probability one $\{M_n\}$ converges to some random variable Y. Incidentally, one can further show that Y has the uniform distribution on $[0,1]$, and this model is equivalent to the simplest case of the famous Pólya urn model.

Solution for Problem 2.11. We have $E(|M_n|) \le E(|M_n|^p)^{1/p} \le B^{1/p} < \infty$ by Jensen's inequality, so $\{M_n : n \ge 0\}$ is also an L^1-bounded martingale. The L^1 convergence theorem tells us that there is an $M_\infty \in L^1$ such that M_n converges almost surely to M_∞, confirming the first assertion of (2.36).

To prove the second assertion, we first note by Fatou's Lemma that

$$E(|M_\infty|^p) = E(\liminf_{n\to\infty} |M_n|^p) \le \liminf_{n\to\infty} E(|M_n|^p) \le B < \infty,$$

so $M_\infty \in L^p$. If we set $D = \sup_n |M_n|$, then $D \in L^p$ by Doob's maximal inequality (2.23), so by the elementary Jensen bound (2.32), we have

$$|M_n - M_\infty|^p \le 2^{p-1}\{|M_n|^p + |M_\infty|^p\} \le 2^{p-1}\{D^p + |M_\infty|^p\}.$$

Since $D^p + |M_\infty|^p \in L^1$, we can therefore apply the dominated convergence theorem to the sequence $|M_n - M_\infty|^p$ to obtain

$$\lim_{n\to\infty} E(|M_n - M_\infty|^p) = E(\lim_{n\to\infty} |M_n - M_\infty|^p) = 0;$$

in other words, $\lim_{n\to\infty} ||M_n - M_\infty||_p = 0$, just as we hoped.

Chapter 3

Solution for Problem 3.1. For part (a), complete the square in the exponent to get

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{tx}e^{-x^2/2}\,dx = e^{t^2/2}\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(x-t)^2/2}\,dx\right\}$$

and observe that the braced integral is equal to one since $e^{-(x-t)^2/2}/\sqrt{2\pi}$ is the density of a Gaussian random variable with mean $\mu = t$ and variance 1.

For (b) one can equate the coefficients in

$$E[e^{tX}] = e^{t^2/2} \quad \Leftrightarrow \quad \sum_{n=0}^{\infty} E(X^n)\frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{t^{2n}}{2^n n!}, \tag{15.41}$$

or one can use integration by parts to get a recursion,

$$\begin{aligned} E(X^{2n}) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^{2n} e^{-x^2/2}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^{2n-1}(xe^{-x^2/2})\, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (2n-1)x^{2n-2} e^{-x^2/2}\, dx = (2n-1)E(X^{2n-2}). \end{aligned}$$

Finally, for (c), one expands, uses the moments, and recognizes the sum:

$$E(e^{itX}) = \sum_{n=0}^{\infty} E(X^n)\frac{i^n t^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2^n n!} = e^{-t^2/2}.$$

For an unlikely (but instructive) alternative, one can also prove (c) by noting

$$f(t) = E(e^{itX}) = \operatorname{Re} E(e^{itX}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} \cos(tx)\, dx,$$

so, by differentiation under the integral and integration by parts, one has

$$\begin{aligned} f'(t) &= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} xe^{-x^2/2} \sin(tx)\, dx \\ &= \frac{-t}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} \cos(tx)\, dx = -tf(t), \end{aligned}$$

from which one deduces that $f(t) = e^{-t^2/2}$ since $e^{-t^2/2}$ is the unique solution of the equation $f'(t) = -tf(t)$ with $f(0) = 1$.

Solution for Problem 3.2. Suppose X is Gaussian, consider an independent U such that $P(U = 1) = 1/2 = P(U = -1)$, and set $Y = UX$.

Solution for Problem 3.3. By the independent increment and Gaussian properties of Brownian motion, the required density is given by

$$f(x_1, x_2, x_3) = \frac{e^{-x_1^2/2t_1}}{\sqrt{2\pi t_1}} \frac{e^{-(x_2-x_1)^2/2(t_2-t_1)}}{\sqrt{2\pi(t_2-t_1)}} \frac{e^{-(x_3-x_2)^2/2(t_3-t_2)}}{\sqrt{2\pi(t_3-t_2)}}. \tag{15.42}$$

Using this density one then formally obtains

$$P(E_S) = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} f(x_1, x_2, x_3)\, dx_1\, dx_2\, dx_3. \tag{15.43}$$

There are times when the explicit densities and integrals such as (15.42) and (15.43) are useful, but in the vast majority of Brownian motion problems they are of limited value. For $n \geq 5$, even the numerical computation of $P(E_S)$ is (close to) infeasible. Thus, in most cases, one is driven to search for indirect methods for computing probabilities.

Solution for Problem 3.4. For the upper bound of (3.25), we modify the argument of Lemma 3.2; specifically, for $x > 0$ we have

$$P(Z \geq x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-u^2/2}\, du \leq \frac{1}{x\sqrt{2\pi}} \int_x^\infty u e^{-u^2/2}\, du = \frac{1}{\sqrt{2\pi}} \frac{e^{-x^2/2}}{x}.$$

For the lower bound, we first note $(u^{-1}e^{-u^2/2})' = -(1+u^{-2})e^{-u^2/2}$, so we have

$$\begin{aligned} P(Z \geq x) &= \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-u^2/2}\, du \geq \frac{1}{\sqrt{2\pi}} \int_x^\infty \frac{1+u^{-2}}{1+x^{-2}} e^{-u^2/2}\, du \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{1+x^{-2}} \frac{e^{-x^2/2}}{x} = \frac{1}{\sqrt{2\pi}} \frac{e^{-x^2/2}}{(x+x^{-1})}. \end{aligned}$$

Finally, the two bounds of (3.26) follow immediately from the observation that $1/\sqrt{2e\pi} \leq e^{-x^2/2}/\sqrt{2\pi} \leq 1/\sqrt{2\pi}$ for all $|x| \leq 1$.

Solution for Problem 3.5. Since $\mathbf{B}_t$ is equal in distribution to $\sqrt{t}\mathbf{B}_1$, the identity (3.27) holds with

$$\begin{aligned} c = E\left(\frac{1}{|\mathbf{B}_1|^2}\right) &= \int_0^\infty \int_0^{2\pi} \int_0^\pi \frac{1}{r^2} \left\{ \frac{1}{(2\pi)^{3/2}} e^{-r^2/2} \right\} r^2 \sin(\varphi)\, d\varphi\, d\theta\, dr \\ &= (2\pi) \frac{1}{(2\pi)^{3/2}} \int_0^\infty e^{-r^2/2}\, dr \int_0^\pi \sin(\varphi)\, d\varphi = (2\pi) \frac{1}{(2\pi)^{3/2}} \left(\frac{1}{2}\sqrt{2\pi} \right) 2 = 1. \end{aligned}$$

Solution for Problem 3.6. For the first part we note that

$$\int_0^1 H_0(x) f(x)\, dx = \int_0^1 H_1(x) f(x)\, dx = \frac{1}{4} \quad \text{and} \quad \int_0^1 H_3(x) f(x)\, dx = \frac{1}{2\sqrt{2}},$$

but all other H_n inner products with f are zero. Thus, as elements of $L^2[0,1]$ we have

$$f(x) = \frac{1}{4} H_0(x) + \frac{1}{4} H_1(x) + \frac{1}{2\sqrt{2}} H_3(x).$$

For second part, we just note (1) each $f \in W$ is trivially in the span of V and (2) each $f \in W$ is in the span of W by direct computation as in the first part.

Solution for Problem 3.7. By the scaling law (3.20) we have

$$\begin{aligned} \tau_A = \min\{t : |B_t| = A\} &\stackrel{\mathrm{d}}{=} \min\{t : \frac{1}{\sqrt{\lambda}} |B_{\lambda t}| = A\} \\ &= \min\{t : |B_{\lambda t}| = A\sqrt{\lambda}\} = \frac{1}{\lambda} \min\{s : |B_s| = A\sqrt{\lambda}\}. \end{aligned}$$

Solution for Problem 3.8. For (a), first note that $\Delta_n(1) = 0$ for $n \geq 1$ and $\Delta_0(t) = t$, so just by substitution one has $B_1 = \lambda_0 Z_0$. From this and a second substitution, one gets

$$U_t \stackrel{\text{def}}{=} B_t - \lambda_0 Z_0 \Delta_0(t) = B_t - tB_1. \tag{15.44}$$

For (b), we take $0 \le s \le t$, apply (15.44), and expand

$$\mathrm{Cov}(U_s, U_t) = E(B_s B_t - sB_1 B_t - tB_s B_1 + stB_1^2) = s - st - st + st = s(1-t).$$

For (c) take $0 \le s \le t \le 1$ and note $\mathrm{Cov}(X_s, X_t) = g(s)g(t)\min(h(s), h(t))$. To limit the search, take $g(s) = s$ and assume that h is monotone decreasing; we would then need $sth(t) = s(1-t)$. By taking $g(s) = s$ and $h(t) = (1-t)/t$, we therefore find that the Gaussian process $X_t = g(t)B_{h(t)}$ has covariance $s(1-t)$, so it is the Brownian bridge.

Finally, for (d), Y_t is a Gaussian process, so we just need to calculate the covariance function. For $0 \le s \le t < \infty$ we have

$$\begin{aligned}\mathrm{Cov}(Y_s, Y_t) &= E[(1+s)U_{s/(1+s)}(1+t)U_{t/(1+t)}] \\ &= (1+s)(1+t)\{s/(1+s)\}\{1 - t/(1+t)\} = s = \min(s,t).\end{aligned}$$

Chapter 4

Solution for Problem 4.1. For the first part, take $\epsilon > 0$ and note that $A = \{\omega : U - V > \epsilon\}$ is $\mathcal{G}$-measurable. The hypothesis then gives us that $E((U-V)\mathbb{I}_A) = 0$, but $E((U-V)\mathbb{I}_A) \ge \epsilon P(A)$ so we must have $P(A) = 0$; that is $P(U > V + \epsilon) = 0$. By symmetry, we also have $P(V > U + \epsilon) = 0$, so $P(|U - V| \le \epsilon) = 1$ for all $\epsilon > 0$, and consequently, $P(U = V) = 1$. For the deduction, take $A \in \mathcal{G}$ and note that the definition of conditional expectation tells us that Y_1 and Y_2 are $\mathcal{G}$-measurable, $E(Y_1\mathbb{I}_A) = E(X\mathbb{I}_A)$, and $E(Y_2\mathbb{I}_A) = E(X\mathbb{I}_A)$; hence, $E(Y_1\mathbb{I}_A) = E(Y_2\mathbb{I}_A)$ for all $A \in \mathcal{G}$ and the first part applies.

Solution for Problem 4.2. For part (a) set $Z = E(X \mid \mathcal{G})$ and note that since $\Omega \in \mathcal{G}$ the definition of $E(X \mid \mathcal{G})$ implies $E(Z\mathbb{I}_\Omega) = E(X\mathbb{I}_\Omega)$. Thus, since $E(Z\mathbb{I}_\Omega) = E(Z)$ and since $E(X\mathbb{I}_\Omega) = E(X)$, we have our target identity (4.26).

For part (b) let $Z_1 = \mathbb{I}_A E(X \mid \mathcal{G})$, $Z_2 = E(\mathbb{I}_A X \mid \mathcal{G})$, and take $B \in \mathcal{G}$. We have $E(\mathbb{I}_B Z_1) = E(\mathbb{I}_B \mathbb{I}_A X) = E(\mathbb{I}_{A\cap B} X)$ and

$$\begin{aligned}E(\mathbb{I}_B Z_2) &= E\big(\mathbb{I}_B E(\mathbb{I}_A X \mid \mathcal{G})\big) \\ &= E(\mathbb{I}_B\{\mathbb{I}_A X\}) && \text{by definition of } E(\mathbb{I}_A X \mid \mathcal{G}) \\ &= E(\mathbb{I}_B \mathbb{I}_A X) = E(\mathbb{I}_{A\cap B} X) && .\end{aligned}$$

Thus, we have $E(\mathbb{I}_B Z_1) = E(\mathbb{I}_B Z_2)$ for all $B \in \mathcal{G}$ so by the uniqueness lemma of Exercise 4.1, we have $Z_1 = Z_2$ with probability one.

By part (b) and the linearity of the conditional expectation, we automatically get the identity (4.28) for linear combinations of indicator functions. In particular, if we let

$$Y_n \stackrel{\text{def}}{=} \sum_{k=-n^2}^{n^2} (k/n)\mathbb{I}(k/n \le Y < (k+1)/n),$$

then since each of the sets $\{k/n \le Y < (k+1)/n\}$ is in $\mathcal{G}$ we have

$$E(XY_n|\mathcal{G}) = Y_n E(X|\mathcal{G}) \quad \text{for all } n = 1, 2, ... \tag{15.45}$$

Since Y_n is bounded, there is a constant C such that $|Y_n(\omega)| \le C$ for all ω and $|XY_n| \le C|X| \in L^1$, so the sequence $\{XY_n\}$ is L^1-dominated. By construction, Y_n converges to Y for all ω, so we can let $n \to \infty$ in the identity (15.45) and apply the dominated convergence theorem to complete the proof of the factorization identity (4.28).

Solution for Problem 4.3. First let $Z_1 = E(X \mid \mathcal{H})$ and $Z_2 = E(E(X \mid \mathcal{G}) \mid \mathcal{H})$. Both Z_1 and Z_2 are $\mathcal{H}$ measurable, so to prove they are equal it suffices to show that $E[\mathbb{I}_A Z_1] = E[\mathbb{I}_A Z_2]$ for all $A \in \mathcal{H}$. For $A \in \mathcal{H}$ the definition of $E(\cdot \mid \mathcal{H})$ gives us $E(\mathbb{I}_A Z_1) = E(\mathbb{I}_A X)$, so now we calculate

$$\begin{aligned}
E[\mathbb{I}_A Z_2] &= E[\mathbb{I}_A E(X \mid \mathcal{G})] && \text{by definition of } E(\cdot \mid \mathcal{H}) \\
&= E[E(X\mathbb{I}_A \mid \mathcal{G})] && \text{since } A \in \mathcal{H} \subset \mathcal{G} \text{ (Factorization Property)} \\
&= E[\mathbb{I}_\Omega E(X\mathbb{I}_A \mid \mathcal{G})] && \text{by definition of } \Omega \\
&= E[\mathbb{I}_\Omega X\mathbb{I}_A] = E[X\mathbb{I}_A] && \text{by definition of } E(\cdot \mid \mathcal{G}) \text{ and } \Omega \in \mathcal{H}.
\end{aligned}$$

Here one should note the need for the third equality; since $\Omega \in \mathcal{G}$ this step sets up the use of the definition of an appeal to the definition of $E(\cdot \mid \mathcal{G})$]. Without some such appeal to this definition, one has very likely "cheated."

Solution for Problem 4.4. We can define f_n on $[0, 1)$ by setting

$$f_n(x) = 2^n \int_{i2^{-n}}^{(i+1)2^{-n}} f(u)\, du \quad \text{for all } x \in [i2^{-n}, (i+1)2^{-n}) \text{ and } 0 \le i < 2^n,$$

alternatively, for $x \in [i2^{-n}, (i+1)2^{-n})$ we have that $f_n(x)$ is the average value of f on $[i2^{-n}, (i+1)2^{-n})$. The martingale property follows from the tower property of conditional expectations; specifically, one has

$$E(f_{n+1} \mid \mathcal{F}_n) = E(E(f|\mathcal{F}_{n+1}) \mid \mathcal{F}_n)) = E(f \mid \mathcal{F}_n) = f_n.$$

For the L^2 boundedness, one has by Jensen's inequality that

$$E(f_n^2) = E(\{E(f \mid \mathcal{F}_n)\}^2) \le E(\{E(f^2 \mid \mathcal{F}_n)\}) = E(f^2) < \infty.$$

To prove part (c), for each x and n we let $I(x, n)$ be the interval in the sequence $[i2^{-n}, (i+1)2^{-n})$, $i = 0, 1, ...,$ that contains x. By the explicit formula for f_n, we have $\inf_{u \in I(x,n)} f(u) \le f_n(x) \le \sup_{u \in I(x,n)} f(u)$, but, by the continuity of f, both of these bounds converge to $f(x)$ as $n \to \infty$.

Solution for Problem 4.5. The countable additivity of Q follows from the monotone convergence theorem for the P-expectations. When we expand the definition of Q we find

$$(E_Q[Z])^q = (E[ZX^p]/E[X^p])^q \text{ and } E_Q[Z^q] = E[Z^q X^p]/E[X^p],$$

so, in longhand, Jensen's inequality tells us that

$$\left(\frac{E[ZX^p]}{E[X^p]}\right)^q \le \frac{E[Z^q X^p]}{E[X^p]}. \tag{15.46}$$

To get closer to Hölder's inequality, we would like to set $ZX^p = XY$, and, since Z is completely at our disposal, we simply take $Z = Y/X^{p-1}$. This choice also gives us $Z^p X^q = Y^q$ since $(p-1)q = p$, and we then see that the bound (15.46) immediately reduces to Hölder's inequality.

This proof may not be as succinct as the customary derivation via the real variable inequality $xy \le x^p/p + y^q/q$, but, from the probabilist's point of view, the artificial measure method has a considerable charm. At a minimum, this proof of Hölder's inequality reminds us that each of our probability inequalities may hide treasures that can be brought to light by leveraging the generality that is calmly concealed in the abstract definition of a probability space.

Solution for Problem 4.6. Taking the hint, we have

$$\begin{aligned} P(\tau > kN) &= P(\tau > kN \text{ and } \tau > (k-1)N) = E\big[\mathbb{I}(\tau > kN)\mathbb{I}(\tau > (k-1)N)\big] \\ &= E\big[\mathbb{I}(\tau > (k-1)N)E(\mathbb{I}(\tau > kN) \,|\, \mathcal{F}_{(k-1)N})\big] \\ &\le (1-\epsilon)E\big[\mathbb{I}(\tau > (k-1)N\big] \le (1-\epsilon)^k, \end{aligned}$$

where in the first inequality we used the hypothesis (4.30) and in the second we applied induction. The corollary $E[\tau^p] < \infty$ then follows by the traditional tail bound method (see, for example, page 4).

Solution for Problem 4.7. By Markov's inequality one has

$$P(|X_n - X| \ge \epsilon) = P(|X_n - X|^\alpha \ge \epsilon^\alpha) \le \epsilon^{-\alpha} E(|X_n - X|^\alpha),$$

and part (a) then follows immediately. For (b) note that since $X_n \to X$ in probability we can choose n_k such that

$$P(|X_{n_k} - X| \ge 1/k) \le 2^{-k}.$$

Since these probabilities have a finite sum, the Borel-Cantelli lemma tells us that with probability one $|X_{n_k} - X| \ge 1/k$ for all but a finite set of values of k. Hence X_{n_k} converges to X with probability one. For (c), choose a subsequence m_k such that X_{m_k} converges with probability one to X, then choose a subsequence n_k of m_k such that X_{n_k} converges with probability one to Y. Now, since X_{n_k} also converges with probability one to X, we have that $X = Y$ with probability one.

For (d) we take $\Omega = [0,1]$ and take $P(A)$ equal to the length (or the Lebesgue measure) of the set $A \subset [0,1]$. We then note that any $n \geq 1$ can be written uniquely as $n = 2^j + k$ where $0 \leq k < 2^j$, and we set

$$X_n(\omega) = \begin{cases} 1 & \text{for } \omega \in \left[k/2^j, (k+1)/2^j\right) \\ 0 & \text{otherwise.} \end{cases}$$

If we set $X(\omega) = 0$ for all $\omega \in [0,1]$, then we have $E(|X_n - X|) = 1/2^j$ and $j \to \infty$ as $n \to \infty$, so $E(|X_n - X|) \to 0$ as $n \to \infty$. On the other hand, by drawing the picture, we see that for every $\omega \in [0,1)$, each of the sets $\{n : X_n(\omega) = 1\}$ and $\{n : X_n(\omega) = 0\}$ is infinite, so $\{X_n\}$ fails converge for all ω in a set with probability one. In fact, for all $\omega \in [0,1]$ we have

$$\liminf_{n\to\infty} X_n(\omega) = 0 \quad \text{and} \quad \limsup_{n\to\infty} X_n(\omega) = 0.$$

Solution for Problem 4.8. For $0 < s < t$ the increment $Y_t - Y_s$ is measurable with respect to the σ-field $\mathcal{G} = \sigma\{B_u : 1/t \leq u < 1/s\}$ and Y_s is measurable with respect to $\mathcal{G}' = \sigma\{B_u : 1/s \leq u\}$. Since Brownian motion has independent increments, $\mathcal{G}$ and $\mathcal{G}'$ are independent; hence, $Y_t - Y_s$ and Y_s are independent. Thus, $\{Y_t : 0 \leq t < \infty\}$ is a process with mean zero and independent increments, and consequently $\{Y_t : 0 \leq t < \infty\}$ is a martingale. For part (b) one just expands the definition of Y_t and uses $E(B_{1/s}B_{1/t} = 1/t)$ for $s < t$. For part (c) the solution, we use an important device; specifically, we exploit the idea that a continuous process may be studied by examining what happens along a countable sequence $\{t_k\}$ along with what happens in the associated "gaps" $[t_{k+1}, t_k]$, $1 \leq k < \infty$.

For any $\epsilon > 0$, we have $P(|X_t| > \epsilon) = P(X_t^2 > \epsilon^2) \leq \epsilon^{-2} t$ by Markov's inequality, so if we take $t_k = 2^{-k}$, then the Borel-Cantelli lemma gives us $P(\limsup_{k\to\infty} |X_{t_k}| > \epsilon) = 0$. For $s > 0$ the process $M_t = X_{t+s} - X_s$ is a martingale, so for $\Delta(s,t) \stackrel{\text{def}}{=} \sup_{s \leq u \leq t} |X_u - X - s|$ we have

$$\begin{aligned} P(\Delta(s,t) > \epsilon) \leq P(\Delta^2(s,t) > \epsilon^2) &\leq \epsilon^{-2} E(\Delta^2(s,t)) \qquad &\text{(Markov's inequality)} \\ &\leq \epsilon^{-2} 2E(|X_t - X_s|^2) = \epsilon^{-2} 2(t-s) \qquad &\text{(Doob's } L^2 \text{ inequality).} \end{aligned}$$

If we set $\Delta_k = \Delta(t_{k+1}, t_k)$, then we have by the Borel-Cantelli lemma that $P(\limsup_{k\to\infty} \Delta_k > \epsilon) = 0$. Finally, we note

$$\{\limsup_{t\to 0} |X_t| > 2\epsilon) \subset \{\limsup_{k\to\infty} |X_{t_k}| > \epsilon\} \cup \{\limsup_{k\to\infty} |\Delta_k| > \epsilon\},$$

so by our results for Δ_k and X_{t_k} we see that X_t converges almost surely to zero as $t \to 0$.

Solution for Problem 4.9. By Doob's stopping time theorem, $M_t = X_{t\wedge\tau}$ is a martingale, so for all t we have $1 = E(M_0) = E(M_t)$. We have $M_t \to X_\tau$ as $t \to \infty$ since τ is almost surely finite, and M_t is bounded above by $e^{\alpha A}$ and bounded below by 0, so by the DCT we have

$$1 = \lim_{t\to\infty} E(M_t) = E(\lim_{t\to\infty} M_t) = E(X_\tau).$$

Now, by symmetry, X_τ is equal to $\exp(\alpha A - \alpha^2\tau/2)$ with probability one-half and equal to $\exp(-\alpha A - \alpha^2\tau/2)$ with probability one-half. Hence,

$$1 = \frac{1}{2}\exp(\alpha A)E(\exp(-\alpha^2\tau/2)) + \frac{1}{2}\exp(-\alpha A)E(\exp(-\alpha^2\tau/2)),$$

so since $\cosh(x) = (e^x + e^{-x})/2$ we have

$$E(\exp(-\alpha^2\tau/2)) = 1/\cosh(\alpha A), \text{ or } \phi(\lambda) = E(\exp(-\lambda\tau)) = 1/\cosh(A\sqrt{2\lambda}).$$

Now we just need to calculate $\phi''(0) = E(\tau^2)$, but it is tedious to differentiate this ratio twice. Instead, one can look up the Taylor series

$$\frac{1}{\cosh x} = 1 - \frac{x^2}{2} + \frac{5x^4}{24} - \cdots \quad \text{or} \quad \frac{1}{\cosh A\sqrt{2\lambda}} = 1 - A^2\lambda + \frac{5}{6}A^4\lambda^2 - \cdots,$$

from which we find $\phi''(0) = E(\tau^2) = \frac{5}{3}A^4$.

The problem we face when we consider the nonsymmetric ruin problem $[A, -B]$ is that the probability that we hit A or $-B$ will depend on τ. Later, in Exercise 8.10, page 173, we will see that one can circumvent this problem by making use of *two* martingales!

Solution for Problem 4.10. The candidate (4.32) for $a(x)$ clearly satisfies $a(x) \to \infty$ as $x \to \infty$, so it just remains to show the convergence of the integral (4.33). From the definition of x_k we have

$$\int_{x_k}^{\infty} P(|X| \geq x)\,dx = 2^{-k} \quad \text{for } k \geq 1,$$

so for $x \in [x_k, x_{k+1}]$ we have $a(x) \leq (2^{-(k+1)})^{-\alpha} = 2^{\alpha(k+1)}$ and hence

$$\begin{aligned}\sum_{k=0}^{\infty}\int_{x_k}^{x_{k+1}} a(x)P(|X| \geq x)\,dx &\leq \sum_{k=0}^{\infty} 2^{\alpha(k+1)}\int_{x_k}^{x_{k+1}} P(|X| \geq x)\,dx \\ &\leq \sum_{k=0}^{\infty} 2^{\alpha(k+1)}2^{-k} < \infty.\end{aligned}$$

Solution for Problem 4.11. The proof is almost immediate. By Lemma 4.5 we can choose a convex ϕ such that $\phi(x)/x \to \infty$ as $x \to \infty$ and

$$E[\phi(|Z|)] < \infty;$$

so, from Jensen's inequality, we have

$$E[\phi(|E(Z|\mathcal{G})|)] \leq E((E(\phi(|Z|)|\mathcal{G})) \leq E(\phi(|Z|)) < \infty.$$

Now, by Lemma 4.4, the last bound is all we need to establish the required uniform integrability.

Solution for Problem 4.12. Since $E(X_t) = 1$ for all $t \geq 0$ and $X_t \geq 0$ we see that $\{X_t : 0 \leq t < \infty\}$ is uniformly bounded in L^1. By the law of large numbers, B_t/t converges almost surely to zero as $t \to \infty$, so the process $X_t = \exp(-t\{\frac{1}{2} - B_t/t\})$ also converge almost surely to zero as $t \to \infty$. On the other hand, since $E(X_t) = 1$ for all $t \geq 0$, so X_t certainly does not converge to 0 in L^1. Therefore, the process $\{X_t : 0 \leq t < \infty\}$ is not uniformly integrable, or else we would have a contradiction to Lemma 4.1, page 69.

Solution for Problem 4.13. For $Y_n = X_{n\wedge\tau}$, Doob's maximal inequality gives $P(X_n \neq Y_n \text{ for some } n = 1, 2, ...) = P(\sup_n |X_n| > \lambda) \leq B/\lambda$, so setting $\lambda = B/\epsilon$ gives us property number three. Taking $Z = \lambda + |X_\tau|\mathbb{I}(\tau < \infty)$ then gives $|Y_n| \leq Z$ for all n, and to show $E(Z) < \infty$ we just note

$$\begin{aligned} E[|X_\tau|\mathbb{I}(\tau < \infty)] &= E[\lim_{n\to\infty} |X_{n\wedge\tau}|\mathbb{I}(\tau < \infty)] \\ &\leq \liminf_{n\to\infty} E[|X_{n\wedge\tau}|\mathbb{I}(\tau < \infty)] \\ &\leq \sup_n E[|X_{n\wedge\tau}|] \leq \sup_n E[|X_n|] \leq B, \end{aligned}$$

where the first inequality used Fatou's lemma, the second was trivial, and the third used the bound $E[|X_{n\wedge\tau}|] \leq E[|X_n|]$ which follows from the fact that $|X_n|$ is a submartingale.

Solution for Problem 4.14. We first note that $\{M_n : n = 0, 1, 2, \ldots\}$ is a discrete-time martingale, so by Exercise 2.11 there is a random variable M_∞ such that M_n converges to M_∞ with probability one and in $L^p(dP)$. Also, for all integers $m \geq 0$ and all real $t \geq m$, we have the trivial bound

$$|M_t - M_\infty| \leq |M_m - M_\infty| + \sup_{\{t: m \leq t < \infty\}} |M_t - M_m|, \tag{15.47}$$

and we already know that $|M_m - M_\infty| \to 0$ with probability one, so (15.47) gives us

$$\limsup_{t\to\infty} |M_t - M_\infty| \leq \lim_{m\to\infty} \sup_{\{t: m \leq t < \infty\}} |M_t - M_m|. \tag{15.48}$$

To estimate the last term, we note that $\{M_t - M_n : n \leq t < \infty\}$ is a continuous martingale, so our freshly-minted maximal inequality (4.17) tells us that for $\lambda > 0$ we have

$$P\big(\sup_{\{t: m \leq t \leq n\}} |M_t - M_m| > \lambda\big) \leq \lambda^{-p} E(|M_n - M_m|^p).$$

Now, since M_n converges to M_∞ in $L^p(dP)$, we just let $n \to \infty$ in the last inequality to find

$$P\Big(\sup_{\{t:m\le t<\infty\}}|M_t-M_m|>\lambda\Big)\le\lambda^{-p}E(|M_\infty-M_m|^p). \tag{15.49}$$

If we let $m\to\infty$ in the bound (15.49), then the L^p convergence of M_m implies that the right-hand side converges to zero and the dominated convergence theorem tells us that we can take the limit inside the probability, so we find

$$P\Big(\lim_{m\to\infty}\sup_{\{t:m\le t<\infty\}}|M_t-M_m|>\lambda\Big)=0. \tag{15.50}$$

Finally, we see from (15.48) and (15.50) that $M_t\to M_\infty$ with probability one, thus completing the proof of the first assertion of (4.35).

The second assertion of (4.35) is proved by a similar, but simpler, argument. We first note that for all m we have

$$||M_t-M_\infty||_p\le||M_m-M_\infty||_p+||M_t-M_m||_p.$$

Since $S_t=|M_t-M_m|$ is a submartingale, we find that for all $n>t$ we have $||M_t-M_m||_p\le||M_n-M_m||_p$, so we also find for all m we have

$$\limsup_{t\to\infty}||M_t-M_\infty||_p\le||M_m-M_\infty||_p+\sup_{\{n:n\ge m\}}||M_n-M_m||_p.$$

Because M_m converges to M_∞ in L^p, the last two terms go to zero as $m\to\infty$, establishing the L^p convergence of M_t.

Solution for Problem 4.15. If we let $\tau_n=\inf\{t\colon |M_t|\ge n\}$, then by Doob's stopping time theorem the process $\{M_{t\wedge\tau_n}:0\le t<\infty\}$ is a martingale for each $n\in\mathbb{N}$. By the definition of τ_n and the continuity of M_t, we have that $|M_{t\wedge\tau_n}|\le n$ for all $0\le t<\infty$. In particular $\{M_{t\wedge\tau_n}:0\le t<\infty\}$ is L^2 bounded, so, by Problem 4.14, we see that $M_{t\wedge\tau_n}$ converges with probability one as $t\to\infty$. Since $M_t(\omega)=M_{t\wedge\tau_n}(\omega)$ for all t if $\tau_n(\omega)=\infty$, we see that as $t\to\infty$ the process $M_t(\omega)$ converges for almost all $\omega\in T_n\equiv\{\omega\colon\tau_n(\omega)=\infty\}$.

Now, by Doob's maximal inequality (4.18) applied to the submartingale $|M_t|$, we have for all T that

$$P\Big(\sup_{0\le t\le T}|M_t|\ge\lambda\Big)\le E(|M_T|)/\lambda\le B/\lambda;$$

so, by letting $T\to\infty$, we have $P(\sup_{0\le t<\infty}|M_t|\ge\lambda)\le B/\lambda$. By the definition of τ_n, the last inequality tells us that for all $n\ge1$ we have $P(T_n)=P(\tau_n=\infty)\ge1-B/n$. So, when we take unions, we find that for $T\equiv\cup_{n=1}^\infty T_n$ we have $P(T)=1$. But $M_t(\omega)$ converges for almost all $\omega\in T$, so M_t converges with probability one. If M_∞ denotes the value of this limit, then by Fatou's lemma and the bound $E(|M_t|)\le B$ we have $E(|M_\infty|)\le B$.

Incidentally, a sequence such as $\{\tau_n\}$ is relate to the notion of a a *localizing sequence*. In this particular problem, $\{\tau_n\}$ helps us to *localize* a problem concerning the large set $L^1(dP)$ to one that deals with the smaller set $L^2(dP)\subset L^1(dP)$.