Stochastic Calculus and Financial Applications
Final Take Home Exam (Steele: Fall 2013)

INSTRUCTIONS. You may consult any books or articles that you find useful. If you use a result that is not from our text, attach a copy of the relevant pages from your source. You may use any software, including the internet, Mathematica, Maple, R, S-Plus, MatLab, etc. **Attach any Mathematica (or similar) code that you use.**

- You may **NOT** consult with any other person about these problems. If you have a question, even one that is just about the meaning of a question, please contact me directly rather than consult with a fellow student.
- I may post "bug reports" or clarifications on our web page, and you should regularly check for these.
- You should strive to make your answers as **clear and complete** as possible, but keep in mind that short answers are often the best ones.
- Never, ever, write down anything that you know — or even vaguely suspect — to be false. If you understand that your argument is incomplete or only heuristic, it should be properly labeled as incomplete or heuristic.
- Don't skip steps without at least telling me. If you use Mathematica or a fact from a table, please say so **and document it.** Otherwise, I stare and stare at line n wondering how you got to $n + 1$ in your head while I can't.
- Use anything from anyplace, but do not steal. If you make use of an argument from some source, give credit to the source. If you find the complete (and correct!) solution to a problem in a book or on the internet, just print out the pages and attach them. You will get full credit.
- Begin each new solution on a new page.
- Arrange your solutions in the natural numerical order. If you do not do problem K, then include a self-standing page that says "Problem K was not done."
- Staple your pages neatly with a high-quality stapler with appropriate length and weight to do a clean and secure job.
- **As discussed in class, you MUST use and complete the cover page given at the website. Self-evaluation is hugely valuable.**

GENERAL ADVICE

(1) In your solutions, please do not just write down things that you think are relevant even though they do not add up to an honest solution. Such lists are useful when you are working on a problem, but, if you offer a list as a solution, you prevent yourself from discovering the idea that is missing.

(2) If you can explain **clearly** something that you tried that did not work, this sometimes is worth a few points. Please do not abuse this offer. With experience, one learns that many sensible ideas do not work. Almost by definition, this is what separates a "problem" from an "exercise."

(3) Keep in mind that a good problem requires that one "overcome some objection." What distinguishes a problem from an exercise is that in a good problem a routine plan does not work. The whole point is to go past the place where routine ideas take you. Still, don't shy away from the obvious; **many of the "problems" here are really "exercises."**

(4) If you do something **extra** that is valid, you can get "bonus" points. These special rewards cannot be determined in advance. They are usual small, but they can be substantial — and they do add up.
(5) The most common source of bonus points is for saying something particularly well. Clear, well-organize, solutions are gems. They deserve to be acknowledged.

THE BIG PICTURE

Almost certainly these instructions will seem to be overly detailed to you. It is true that they are detailed, but they evolved case by case. Each rule deals with some previous misunderstanding. When you start teaching (and grading) I encourage you to follow this example. There is no harm in a having a few creative (yet compassionate) eccentricities.

There is a final motivation for this long list of rules and suggestions. **Detailed instructions provide clear coaching for excellence.** We all do wonderfully better when we are lucky enough to know what we need to do. This is the kind of break one seldom gets in research.

Due Date and Place: I need the exam in THREE FORMS. First, a hard copy **with its completed self-evaluation cover sheet** is due in my mail box in JMHH Suite 400 on in my office JMHH 447 on the date given on our web page. Hard copy solutions may be emailed or sent by FedEx. Fax is not acceptable. ALSO: I need the a PDF of your solutions and your latex source (including any auxiliary files). For filenames please use *YOUR NAME*.

NOTE: This is Version 8.1 of the Final. You now have 15 problems and that's the total. Except for possible bug report corrections, this is the FINAL version of the exam.

PROBLEM 1: FEYNMAN-KAC WITH END CONDITIONS

Given real values $a < b$ consider the PDE

$$\begin{aligned} u_t(t,x) &= \frac{1}{2}u_{xx}(t,x) + q(x)u(t,x) \quad t > 0 \\ u(0,x) &= f(x) \quad a < x < b \\ u(t,a) &= u(t,b) = 0 \quad t > 0. \end{aligned}$$

Assume that this PDE has a solution and use the method from class to show that for $a < x < b$ one has the stochastic representation:

$$u(t,x) = E_x\left\{ f(B_t)\mathbb{1}(a < B_s < b \text{ for all } 0 \le s \le t) \exp\left(\int_0^t q(B_s)\,ds\right)\right\}.$$

Consider the specialization of your formula when $f(x) = 1$ and $q(x) = 0$ for all x. Take $x = (a+b)/2$ and evaluate the last expression. For extra credit, see if you can evaluate this expression for $f(x) = 1$, $q(x) = 0$, $a = -\pi$ and $b = \pi$ (or other nice values); check that you do indeed have a solution of the PDE in this special case. Note: For the last piece fell free to poke around in books for anything that might help.

PROBLEM 2: WARM-UP PROBLEMS

(a) Take $f(x) = x^k$ and use Itô's formula to get a recursion that permits you to calculate $E[B_t^k]$ for all t and all $k = 0, 1, 2, \ldots$.

(b) Take $f(x) = e^{\alpha x}$ and use Itô's formula to get an ODE that permits you to calculate the moment generating function of B_t. Use this formula to check your answer to part (a).

(c) Suppose that B_t and B'_t are two independent Brownian motions. From elementary probability theory you know that $X_t = (B_t)^2 + (B'_t)^2$ has the exponential distribution. Show that one can deduce this from Itô's formula. You may want to look for an ODE for the characteristic function. For extra credit, can you get some other piece of classical distribution theory from Itô's formula?

PROBLEM 3: TWO TRIGONOMETRIC MOTIONS

Consider the processes defined by

$$W_t = \int_0^t \cos(u)\, dB_u \qquad \text{and} \qquad \hat{W}_t = \int_0^t \sin(u)\, dB_u.$$

(a) Prove or disprove the "Pythagorean Identity": $E(W_t^2 + \hat{W}_t^2) = t$.

(b) Find the set of all values t such that the processes W_t and $\hat{W}_t$ are uncorrelated.

(c) Prove or disprove that $M_t = W_t^2 + \hat{W}_t^2 - t$ is an honest martingale.

(d) Give conditions on $f(t, x, y)$ that suffice for $f(t, W_t, \hat{W}_t)$ be a local martingale.

PROBLEM 4. TWO RELATED PROCESSES DEFINED BY INTEGRALS

Consider the processes defined by *definite* integral

$$X_t = \sqrt{2}\, t^2 \int_0^\infty B_u e^{-ut}\, du$$

and the *definite* stochastic integral

$$Y_t = \sqrt{2}\, t \int_0^\infty e^{-ut}\, dB_u.$$

(a) Calculate $E(X_t)$, $E(X_t^2)$, $E(Y_t)$, and $E(Y_t^2)$.

(b) Are the processes $\{X_t\}$ and $\{Y_t\}$ are equivalent? In other words, does one have for any $0 \le t_1 < t_2 < \cdots t_n$ that the vector $(X_{t_1}, X_{t_2}, ..., X_{t_n})$ has the same distribution as the vector $(Y_{t_1}, Y_{t_2}, ..., Y_{t_n})$.

(c) How about the two processes $Z_t = X_{at}$ and $Z'_t = \sqrt{a}X_t$; are they equivalent?

PROBLEM 5. EXPLORATION OF A PRODUCT PROCESS

Consider the process defined by

$$X_t = B_t^2 B'_t - \int_0^t B'_s\, ds.$$

where B_t and B'_t are two independent Brownian motions.

(a) Prove or disprove that X_t is an honest martingale.

(b) The quadratic variation $\langle Z \rangle_t$ of the process Z_t is a random variable, if the quadratic variation exists. Calculate the expected value of that random variable, i.e. calculate $E[\langle X \rangle_t]$, where X_t is the process defined in part (a).

(c) It's immediate that X_t and B_t are uncorrelated. What is the covariance of X_t and B'_t?

PROBLEM 6. SOLUTION OF AN SDE

Solve the following SDE:

$$dX_t = 2X_t(1 - X_t)\,dt + X_t\,dB_t \quad \text{where } 0 < X_0 = x_0 < 1.$$

Here a "solution" is an expression for X_t as a function of Brownian motion and related processes such as integrals of Brownian motion.

PROBLEM 7. IF IT QUACKS LIKE A DUCK

Suppose that $\{M_t : 0 \leq t < \infty\}$ is a local martingale with respect to the filtration of Brownian motion. Suppose that the process $Y_t = M_t^2 - t$ is also a local martingale. Further, assume that $M_0 = 0$.

(a) Prove that M_t is normally distributed with mean zero and variance t. Itô's formula should get you going. You can assume throughout the problem that M_t is a "standard process" so that our general version of Itô's formula may be applied.

(b) Refine your argument for part (a) to show more precisely that for all $0 \leq s \leq t$ one has

$$E(\exp(i\theta(M_t - M_s) \,|\, \mathcal{F}_s) = \exp(-\theta^2(t - s)/2),$$

and explain why this implies that M_t is actually Brownian motion.

PROBLEM 8. THE SIMPLEST COMPLEX MARTINGALE

Let $\{Z_t : t \geq 0\}$ be complex Brownian motion (i.e. $Z_t = X_t + iY_t$) where the two processes $\{X_t : t \geq 0\}$ and $\{Y_t : t \geq 0\}$) are independent Brownian motions. Let $Z_0 = i$ (i.e. $X_0 = 0$ and $Y_0 = 1$).

(a) Show that for any $\lambda \in \mathbb{R}$, the process $\{\exp(i\lambda Z_t) : t \geq 0\}$ is a martingale.

(b) Let τ be the first time that the process $\{Z_t : t \geq 0\}$ hits the real axis. Repeat the familiar dance with Doob's stopping time theorem to show that one has

$$E[\exp(i\lambda Z_\tau)] = e^{-\lambda} \quad \text{for } \lambda \geq 0 \tag{1}$$

(c) Use the last observation to argue that X_τ has a Cauchy distribution. You may have to refresh your memory of the characteristic function of a Cauchy distribution.
(d) Look back over your argument. Doesn't the martingale argument imply that (1) holds for all $\lambda \in \mathbb{R}$? This looks like a sound inference, but for negative λ the right side of (1) is bigger than one, which is not the kind of behavior one can have for a characteristic function. Resolve this paradox.

PROBLEM 9. FOR EXAMPLE ...

Give an example of stopping times S and T such that $P(S < T) = 1$ and such that $E(S) < \infty$, yet nevertheless one has

$$E(B_T^2) < E(B_S^2).$$

To nail down the point of this example, recall that $M_t = B_t^2 - t$ is a martingale so an overly naive application of Doob's stopping time theorem might suggest that one would have $E(B_S^2) < E(B_T^2)$. Explain why Doob's stopping time theorem does not imply the last identity for your S and T.

Problem 10. Real Soup from Stochastic Vegetables

If $f : [0,1] \to \mathbb{R}$ is a measurable function, then Hardy's inequality is the bound

$$\int_0^1 \left(\frac{1}{t} \int_0^t f(x)\, dx \right)^2 dt \leq 4 \int_0^1 f^2(x)\, dx.$$

Consider the probability space $(\Omega, \mathcal{F}, P)$ given by taking $\Omega = [0,1]$, $\mathcal{F}$ the Borel sets, and P the uniform measure. Let $\mathcal{F}_t$ be the decreasing sequence of σ-fields defined by

$$\mathcal{F}_t = \sigma\left\{[0,t], \boldsymbol{B}([t,1])\right\}$$

where $\boldsymbol{B}([t,1])$ is the set of all Borel subsets of $[t,1]$.

(a) Check that $M_t = E[f \mid \mathcal{F}_{1-t}]$ is a martingale.

(b) Use the preceding observation and what you know about martingales to prove Hardy's inequality. This will be easy once you have drawn for yourself a proper picture of the function $s \mapsto E[f \mid \mathcal{F}_s]$.

(c) Does your proof suggest an L^p version of Hardy's inequality? What replaces the constant 4?

Problem 11. Using What We Know

(a) Let $X_t = \arctan(B_t)$, $0 \leq t < \infty$, and explain why one has the representation

$$X_t = \int_0^t \frac{1}{1+B_s^2}\, dB_s - \int_0^t \frac{B_s}{(1+B_s^2)^2}\, ds.$$

(b) Explain how Doob's maximal inequality implies

$$E \sup_{0 \leq t \leq T} \left| \int_0^t \frac{1}{1+B_s^2}\, dB_s \right| \leq 2T^{1/2}.$$

Here you may need some preliminary step to "set up" the application of Doob's inequality.

(c) Show by bounding the integrand that

$$E \sup_{0 \leq t \leq T} \left| \int_0^t \frac{B_s}{(1+B_s^2)^2}\, ds \right| \leq \frac{T3\sqrt{3}}{16}.$$

(d) Each of these bounds seems extravagant since $|X_t| \leq \pi/2$ with probability one for all $t > 0$. Can you use this powerful information about X_t to improve either of the two integral estimates, or is X_t kept so small by some cancelation between its two summands? It would be particularly nice if you could show that one of these integrals is bounded by a constant. We could then conclude that both are bounded by constants. Alternatively if you can you show that one of these expectations goes to infinity as $T \to \infty$; this would suggest that the there is indeed some magical cancelation in our representation for X_t.

PROBLEM 12. TIME SCALING OF A POLYNOMIAL

Let $P(x)$ be a degree n polynomial, and assume that a and b are real constants such that the process $X_t = t^a P(t^{-b} B_t)$ is a martingale for all $t \geq 0$.

- Use Itô's formula to find a second order ordinary differential equation that must be satisfied by P.
- Determine the values of the constants a and b.
- If $n = 3$ and we have a polynomial of the form, $P(x) = x^3 + c_2 x^2 + c_1 x + c_0$ determine the values c_0, c_1, and c_2.

PROBLEM 13. A SCREWY PROBLEM

Consider a vector process $V_t = (X_t, Y_t, Z_t)$ in $\mathbb{R}^3$ that satisfies the stochastic system:

$$\begin{aligned} dX_t &= -\frac{1}{2} X_t\, dt - Y_t\, dB_t \\ dY_t &= -\frac{1}{2} Y_t\, dt + X_t\, dB_t \\ dZ_t &= dt. \end{aligned}$$

To start the process take $V_t = (1, 0, 0)$.

(a) Show that $X_t^2 + Y_t^2 = 1$ for all $t \geq 0$ and describe the kind of surface in $\mathbb{R}^3$ on which V_t lives.

(b) Let $\tau = \min\{t : (X_t, Y_t) = (1/\sqrt{2}, 1/\sqrt{2}) \quad \text{or} \quad (X_t, Y_t) = (-1, 0)\}$. Calculate the value of the probability

$$P((X_\tau, Y_\tau) = (-1, 0)).$$

There are several ways to do this, but you may find polar coordinates to be useful.

(c) Calculate $E(X_\tau Y_\tau)$.

PROBLEM 14. OUTRAGEOUS CLAIMS

Consider a traditional stock and bond model:

$$dS_t = \mu S_t\, dt + \sigma S_t dB_t \quad \text{and} \quad d\beta_t = r\beta_t\, dt,$$

where $\mu > 0$, $\sigma > 0$ and $r > 0$ are constants.

(a) Consider a time interval $[0, T]$ and a contingent claim that pays $2S_T$ at time T, so we have $h(x) = 2x$ in the Black-Scholes formulation or we have $X = 2S_T$ in the Harrison-Kreps formulation of the theory of contingent claims. Use your gambler's good sense to guess the arbitrage value at time $t \in [0, T]$ of this contingent claim. Next, use the Black-Scholes PDE to justify your guess, i.e. show that your formula for the claim value satisfies the PDE. Also, check that in this case the solution of the BS-PDE is unique. For the last step you can quote any results from the text may help.

(b) Now give a second justification of your guess using the Harrison-Kreps formulation of the theory of contingent claim pricing. Specifically, find the measure Q that appears in the Harrison-Kreps formula and work out the value of the conditional expectation, etc. You can anticipate that the integrals you will encounter will be quite simple. Have faith.

(c) Next, consider a strange world where volatility is very low and interest rates are negative (Switzerland?). Suppose moreover that one has $r = -\sigma^2$. Determine the arbitrage value at time $t \in [0, T]$ of a contingent claim that pays S_T^2 at time T.

You can uses either the BS method or the HK method, or both if you are feeling energetic.

(d) Finally, you generalize the curious result of part (c) in some sensible way? For example, are there conditions on σ and r that would make it easy to value a contingent claim that pays S_T^α at time T where $\alpha \in (0, \infty)$.

Problem 15. Nibbling Around Krylov (2002)

(a) Krylov (2002, page 3) uses a version of Doob's maximal inequality for local submartingales. Here we consider a special case. Suppose that M_t, $0 \le t < \infty$, is a **LOCAL** martingale. Prove — as Krylov asserts more generally — that one has

$$E\{\sup_{0\le t\le T} M_t^2\} \le 4 \sup_{0\le t\le T} E(M_t^2).$$

For extra credit, discuss this further. For example, you might prove the version that Krylov uses. Finally, can one replace the inequality with a simpler one, such as one without a supremum on the right side?

(b) Paralleling Krylov (2002, page 2), consider the process $M_t = 2^{-1} B_{t\wedge\tau}$ where τ is the first time that the Brownian motion exits the interval $[-\pi, \pi]$. Let $\langle M\rangle$ be the quadratic variation of M_t, $0 \le t < \infty$, over all $[0, \infty)$. Krylov asserts that

$$E[\exp(\frac{1}{2}\langle M\rangle)] = \infty.$$

Explain why this is true. Note: There are some typos in this part of Krylov's paper.

(c) Krylov (2002, page 3) says at the end of the first paragraph, "In like manner (6) is proved". Complete this thought. Specifically, use good, well-explained notation to write down what (6) asserts and then provide the omitted details of the argument. Feel free to be generously clear with your explanation.

Envoi

I hope that these problems are interesting to you. They should be accessible to everyone, though some may be challenging. Even if you took a brief "break" from the course, don't count yourself out. You can still do a good job on these problems if that is your desire and if you invest the time to give them a serious try.

Calculate boldly, but then look back over your work to avoid slips, gaps, or misstatements. Invest some energy in your write up of the solutions; capacity for clear exposition is useful skill to develop.

Don't forget that bonus points are available for every problem. If you think of something further that can be said about one of the problems, by all means tell me what you have discovered. This works best if you first give the "plain vanilla" solution then follow up with your additional discoveries.

Good luck to all!