10

Hilbert's Inequality and Compensating Difficulties

Some of the most satisfying experiences in problem solving take place when one starts out on a natural path and then bumps into an unexpected difficulty. On occasion this deeper view of the problem forces us to look for an entirely new approach. Perhaps more often we only need to find a way to press harder on an appropriate variation of the original plan.

This chapter's introductory problem provides an instructive case; here we will discover *two difficulties*. Nevertheless, we manage to achieve our goal by pitting one difficulty against the other.

Problem 10.1 (Hilbert's Inequality)

Show that there is a constant C such that for every pair of sequences of real numbers $\{a_n\}$ and $\{b_n\}$ one has

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < C\bigg(\sum_{m=1}^{\infty} a_m^2\bigg)^{\frac{1}{2}}\bigg(\sum_{n=1}^{\infty} b_n^2\bigg)^{\frac{1}{2}}. \tag{10.1}$$

Some Historical Background

This famous inequality was discovered in the early 1900s by David Hilbert; specifically, Hilbert proved that the inequality (10.1) holds with $C = 2\pi$. Several years after Hilbert's discovery, Issai Schur provided a new proof which showed Hilbert's inequality actually holds with $C = \pi$. We will see shortly that no smaller value of C will suffice.

Despite the similarities between Hilbert's inequality and Cauchy's inequality, Hilbert's original proof did not call on Cauchy's inequality; he took an entirely different approach that exploited the evaluation of some cleverly chosen trigonometric integrals. Nevertheless, one can prove

Hilbert's inequality through an appropriate application of Cauchy's inequality. The proof turns out to be both simple and instructive.

If S is any countable set and $\{\alpha_s\}$ and $\{\beta_s\}$ are collections of real numbers indexed by S, then Cauchy's inequality can be written as

$$\sum_{s\in S}\alpha_s\beta_s \le \left(\sum_{s\in S}\alpha_s^2\right)^{\frac{1}{2}}\left(\sum_{s\in S}\beta_s^2\right)^{\frac{1}{2}}. \tag{10.2}$$

This modest reformulation of Cauchy's inequality sometimes helps us see the possibilities more clearly, and here, of course, one hopes that wise choices for S, $\{\alpha_s\}$, and $\{\beta_s\}$ will lead us from the bound (10.2) to the Hilbert's inequality (10.1).

An Obvious First Attempt

If we charge ahead without too much thought, we might simply take the index set to be $S = \{(m,n) : m \ge 1, n \ge 1\}$ and take α_s and β_s to be defined by the splitting

$$\alpha_s = \frac{a_m}{\sqrt{m+n}} \quad \text{and} \quad \beta_s = \frac{b_n}{\sqrt{m+n}} \qquad \text{where } s = (m,n).$$

By design, the products $\alpha_s\beta_s$ recapture the terms one finds on the left-hand side of Hilbert's inequality, but the bound one obtains from Cauchy's inequality (10.2) turns out to be disappointing. Specifically, it gives us the double sum estimate

$$\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n}\right)^2 \le \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m^2}{m+n}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{b_n^2}{m+n} \tag{10.3}$$

but, unfortunately, both of the last two factors turn out to be infinite.

The first factor on the right side of the bound (10.3) diverges like a harmonic series when we sum on n, and the second factor diverges like a harmonic series when we sum on m. Thus, in itself, inequality (10.3) is virtually worthless. Nevertheless, if we look more deeply, we soon find that the complementary nature of these failings points the way to a wiser choice of $\{\alpha_s\}$ and $\{\beta_s\}$.

Exploiting Compensating Difficulties

The two sums on the right-hand side of the naive bound (10.3) diverge, but the good news is that they diverge for different reasons. In a sense, the first factor diverges because

$$\alpha_s = \frac{a_m}{\sqrt{m+n}}$$

is too big as a function of n, whereas the second factor diverges because

$$\beta_s = \frac{b_n}{\sqrt{m+n}}$$

is too big as a function of m. All told, this suggests that we might improve on α_s and β_s if we multiply α_s by a decreasing function of n and multiply β_s by a decreasing function of m. Since we want to preserve the basic property that

$$\alpha_s \beta_s = \frac{a_m b_n}{m+n},$$

we may not need long to hit on the idea of introducing a parametric family of candidates such as

$$\alpha_s = \frac{a_m}{\sqrt{m+n}} \left(\frac{m}{n}\right)^{\lambda} \quad \text{and} \quad \beta_s = \frac{b_n}{\sqrt{m+n}} \left(\frac{n}{m}\right)^{\lambda}, \tag{10.4}$$

where $s = (m, n)$ and where $\lambda > 0$ is a constant that can be chosen later. This new family of candidates turns out to lead us quickly to the proof of Hilbert's inequality.

Execution of the Plan

When we apply Cauchy's inequality (10.2) to the pair (10.4), we find

$$\left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} \right)^2 \le \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m^2}{m+n} \left(\frac{m}{n}\right)^{2\lambda} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{b_n^2}{m+n} \left(\frac{n}{m}\right)^{2\lambda},$$

so, when we consider the first factor on the right-hand side we see

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m^2}{m+n} \left(\frac{m}{n}\right)^{2\lambda} = \sum_{m=1}^{\infty} a_m^2 \sum_{n=1}^{\infty} \frac{1}{m+n} \left(\frac{m}{n}\right)^{2\lambda}.$$

By the symmetry of the summands $a_m b_n/(m+n)$ in our target sum, we now see that the proof of Hilbert's inequality will be complete if we can show that for some choice of λ there is a constant $B_\lambda < \infty$ such that

$$\sum_{n=1}^{\infty} \frac{1}{m+n} \left(\frac{m}{n}\right)^{2\lambda} \le B_\lambda \qquad \text{for all } m \ge 1. \tag{10.5}$$

Now we just need to estimate the sum (10.5), and we first recall that for any nonnegative decreasing function $f : [0, \infty) \to \mathbb{R}$, we have the integral bound

$$\sum_{n=1}^{\infty} f(n) \le \int_0^{\infty} f(x)\, dx.$$

In the specific case of $f(x) = m^{2\lambda}x^{2\lambda}(m+x)^{-1}$, we therefore find

$$\sum_{n=1}^{\infty} \frac{1}{m+n}\left(\frac{m}{n}\right)^{2\lambda} \leq \int_0^{\infty} \frac{1}{m+x}\frac{m^{2\lambda}}{x^{2\lambda}}\,dx = \int_0^{\infty} \frac{1}{(1+y)}\frac{1}{y^{2\lambda}}\,dy, \quad (10.6)$$

where the last equality comes from the change of variables $x = my$. The integral on the right side of the inequality (10.6) is clearly convergent when λ satisfies $0 < \lambda < 1/2$ and, by our earlier observation (10.5), the existence of any such λ would suffice to complete the proof of Hilbert's inequality (10.1).

Seizing an Opportunity

Our problem has been solved as stated, but we would be derelict in our duties if we did not take a moment to find the value of the constant C that is provided by our proof. When we look over our argument, we actually find that we have proved that Hilbert's inequality (10.1) must hold for any $C = C_\lambda$ with

$$C_\lambda = \int_0^{\infty} \frac{1}{(1+y)}\frac{1}{y^{2\lambda}}\,dy \qquad \text{for } 0 < \lambda < 1/2. \qquad (10.7)$$

Naturally, we should find the value of λ that provides the smallest of these.

By a quick and lazy consultation of *Mathematica* or *Maple*, we discover that we are in luck. The integral for C_λ turns out to both simple and explicit:

$$\int_0^{\infty} \frac{1}{(1+y)}\frac{1}{y^{2\lambda}}\,dy = \frac{\pi}{\sin 2\pi\lambda} \qquad \text{for } 0 < \lambda < 1/2. \qquad (10.8)$$

Now, since $\sin 2\pi\lambda$ is maximized when $\lambda = 1/4$, we see that the smallest value attained by C_λ with $0 < \lambda < 1/2$ is equal to

$$C = C_{1/4} = \int_0^{\infty} \frac{1}{(1+y)}\frac{1}{\sqrt{y}}\,dy = \pi. \qquad (10.9)$$

Quite remarkably, our direct assault on Hilbert's inequality has almost effortlessly provided the sharp constant $C = \pi$ that was discovered by Schur.

This is a fine achievement for Cauchy's inequality, but it should not be oversold. Many proofs of Hilbert's inequality are now available, and some of these are quite brief. Nevertheless, for the connoisseur of techniques for exploiting Cauchy's inequality, this proof of Hilbert's inequality is a sweet victory.

Finally, there is a small point that we should note in passing. The integral (10.8) is actually a textbook classic; both Bak and Newman (1997) and Cartan (1995) use it to illustrate the standard technique for integrating $R(x)/x^{\alpha}$ over $[0, \infty)$ where $R(x)$ is a rational function and $0 < \alpha < 1$. This integral also has a connection to a noteworthy gamma function identity that is described in Exercise 10.8.

Of Miracles and Converses

For a Cauchy–Schwarz argument to be precise enough to show that one can take $C = \pi$ in Hilbert's inequality may seem to require a miracle, but there is another way of looking at the relation between the two sides of Hilbert's inequality that makes it clear that no miracle was required. With the right point of view, one can see that both π and the special integrals (10.8) have an inevitable role. To develop this connection, we will take on the challenge of proving a converse to our first problem.

Problem 10.2 *Suppose that the constant C satisfies*

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} < C \biggl(\sum_{m=1}^{\infty} a_m^2 \biggr)^{\frac{1}{2}} \biggl(\sum_{n=1}^{\infty} b_n^2 \biggr)^{\frac{1}{2}} \tag{10.10}$$

for all pairs of sequences of real numbers $\{a_n\}$ and $\{b_n\}$. Show that $C \geq \pi$.

If we plug any pair of sequences $\{a_n\}$ and $\{b_n\}$ into the inequality (10.10) we will get some lower bound on c, but we will not get too far with this process unless we find some systematic way to guide our choices. What we would really like is a parametric family of pairs $\{a_n(\epsilon)\}$ and $\{b_n(\epsilon)\}$ that provide us with a sequence of lower bounds on C that approach π as $\epsilon \to 0$. This surely sounds good, but how do we find appropriate candidates for $\{a_n(\epsilon)\}$ and $\{b_n(\epsilon)\}$?

Stress Testing an Inequality

Two basic ideas can help us narrow our search. First, we need to be able to calculate (or estimate) the sums that appear in the inequality (10.10). We cannot do many sums, so this definitely limits our search. The second idea is more subtle; we need to put the inequality *under stress.* This general notion has many possible interpretations, but here it at least suggests that we should look for sequences $\{a_n(\epsilon)\}$ and $\{b_n(\epsilon)\}$ such that all the quantities in the inequality (10.10) tend to infinity as $\epsilon \to 0$. This particular strategy for stressing the inequality (10.10) may not seem too compelling when one faces it for the first time, but experience with even a few examples is enough to convince most people that the principle contains more than a drop of wisdom.

Without a doubt, the most natural candidates for $\{a_n(\epsilon)\}$ and $\{b_n(\epsilon)\}$ are given by the identical twins

$$a_n(\epsilon) = b_n(\epsilon) = n^{-\frac{1}{2}-\epsilon}.$$

For this choice, one may easily work out the estimates that are needed to understand the right-hand side of Hilbert's inequality. Specifically, we see that as $\epsilon \to 0$ we have

$$\left(\sum_{m=1}^{\infty} a_m^2(\epsilon)\right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} b_n^2(\epsilon)\right)^{\frac{1}{2}} = \sum_{n=1}^{\infty} \frac{1}{n^{1+2\epsilon}} \sim \int_1^{\infty} \frac{dx}{x^{1+2\epsilon}} = \frac{1}{2\epsilon}. \quad (10.11)$$

Closing the Loop

To complete the solution of Problem 10.2, we only need to show that the corresponding sum for the left-hand side of Hilbert's inequality (10.10) is asymptotic to $\pi/2\epsilon$ as $\epsilon \to 0$. This is indeed the case, and the computation is instructive. We lay out the result as a lemma.

Double Sum Lemma.

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}+\epsilon}} \frac{1}{m^{\frac{1}{2}+\epsilon}} \frac{1}{m+n} \sim \frac{\pi}{2\epsilon} \qquad \text{as } \epsilon \to 0.$$

For the proof, we first note that integral comparisons tell us that it suffices to show

$$I(\epsilon) = \int_1^{\infty} \int_1^{\infty} \frac{1}{x^{\frac{1}{2}+\epsilon}} \frac{1}{y^{\frac{1}{2}+\epsilon}} \frac{1}{x+y} \, dxdy \sim \frac{\pi}{2\epsilon} \qquad \text{as } \epsilon \to 0,$$

and the change of variables $u = y/x$ also tells us that

$$I(\epsilon) = \int_1^{\infty} x^{-1-2\epsilon} \left[\int_{1/x}^{\infty} u^{-\frac{1}{2}-\epsilon} \frac{du}{1+u} \right] dx. \quad (10.12)$$

This integral would be easy to calculate if we could replace the lower limit $1/x$ of the inside integral by 0, and, to estimate how much damage such a change would cause, we first note that

$$0 < \int_0^{1/x} u^{-\frac{1}{2}-\epsilon} \frac{du}{1+u} < \int_0^{1/x} u^{-\frac{1}{2}-\epsilon} \, du = \frac{x^{-\frac{1}{2}+\epsilon}}{\frac{1}{2}-\epsilon}.$$

When we use this bound in equation (10.12) and write the result using big O notation of Landau (say, as defined on page 120), then we find

$$I(\epsilon) = \int_1^\infty x^{-1-2\epsilon}\Big\{\int_0^\infty u^{-\frac{1}{2}-\epsilon}\,\frac{du}{1+u}\Big\}\,dx + O\Big(\int_1^\infty x^{-\frac{3}{2}-\epsilon}\,dx\Big)$$
$$= \frac{1}{2\epsilon}\int_0^\infty u^{-\frac{1}{2}-\epsilon}\,\frac{du}{1+u} + O(1).$$

Finally, for $\epsilon \to 0$, we see from our earlier experience with the integral (10.9) that we have

$$\int_0^\infty u^{-\frac{1}{2}-\epsilon}\,\frac{du}{1+u} \to \int_0^\infty u^{-\frac{1}{2}}\,\frac{du}{1+u} = \pi,$$

so the proof of the lemma is complete.

Finding the Circle in Hilbert's Inequality

Any time π appears in a problem that has no circle in sight, there is a certain sense of mystery. Sometimes this mystery remains without a satisfying resolution, but, in the case of Hilbert's inequality, a geometric explanation for the appearance of π was found in 1993 by Krysztof Oleszkiewicz. This discovery is a bit off of our central theme, but it does build on the calculations we have just completed, and it is too lovely to miss.

Quarter Circle Lemma. For all $m \geq 1$, we have the bound

$$\sum_{n=1}^{\infty} \frac{1}{m+n}\Big(\frac{m}{n}\Big)^{\frac{1}{2}} < \pi. \tag{10.13}$$

For the proof, we first note that the shaded triangle of Figure 10.1 is similar to the triangle T determined by $(0,0)$, $(\sqrt{m}, \sqrt{n-1})$, and $(\sqrt{m}, \sqrt{n})$, and the area of T is simply $\frac{1}{2}\sqrt{m}(\sqrt{n} - \sqrt{n-1})$. Thus, one finds by scaling that the area A_n of the shaded triangle is given by

$$A_n = \Big(\frac{\sqrt{m}}{\sqrt{n+m}}\Big)^2 \frac{1}{2}\sqrt{m}(\sqrt{n} - \sqrt{n-1}). \tag{10.14}$$

Since $1/\sqrt{x}$ is decreasing on $[0, \infty)$, we have

$$\sqrt{n} - \sqrt{n-1} = \frac{1}{2}\int_{n-1}^{n} \frac{dx}{\sqrt{x}} > \frac{1}{2\sqrt{n}}$$

so, in the end, we find

$$A_n > \frac{1}{4}\frac{m}{m+n}\frac{\sqrt{m}}{\sqrt{n}}. \tag{10.15}$$

Finally, what makes this geometric bound most interesting is that all

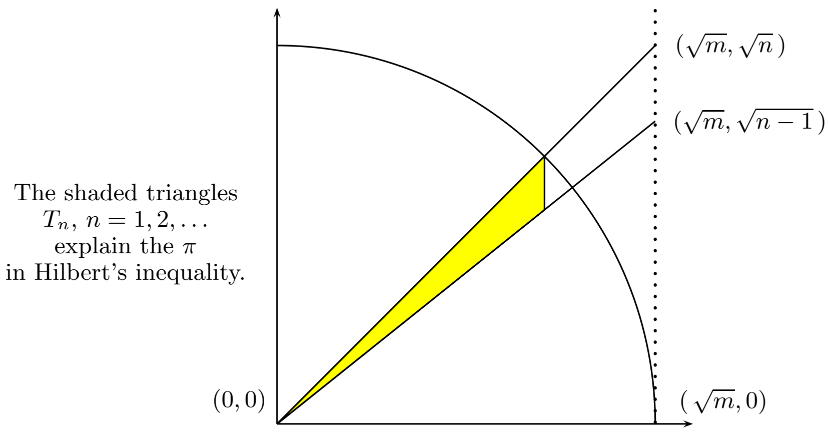

Fig. 10.1. The shaded triangle is similar to the triangle determined by the three points $(0,0)$, $(\sqrt{m}, \sqrt{n-1})$, and $(\sqrt{m}, \sqrt{n})$ so we can determine its area by geometry. Also, the triangles T_n have disjoint interiors so the sum of their areas cannot exceed $\pi/4$. These facts give us the proof of the Quarter Circle Lemma.

of the shaded triangles are contained in the quarter circle. They have disjoint interiors, so we find that the sum of their areas is bounded by $\pi m/4$, the area of the quarter circle with radius $\sqrt{m}$ that contains them.

Exercises

Exercise 10.1 (Guaranteed Positivity)

Show that for any real numbers $a_1, a_2, \ldots, a_n$ one has

$$\sum_{j,k=1}^{n} \frac{a_j a_k}{j+k} \geq 0 \tag{10.16}$$

and, more generally, show that for positive $\lambda_1, \lambda_2, \ldots, \lambda_n$ one has

$$\sum_{j,k=1}^{n} \frac{a_j a_k}{\lambda_j + \lambda_k} \geq 0. \tag{10.17}$$

Obviously the second inequality implies the first, so the bound (10.16) is mainly a hint which makes the link to Hilbert's inequality. As a better hint, one might consider the possibility of representing $1/\lambda_j$ as an integral.

Exercise 10.2 (Insertion of a Fudge Factor)

There are many ways to continue the theme of Exercise 10.1, and this exercise is one of the most useful. It provides a generic way to leverage an inequality such as Hilbert's.

Show that if the complex array $\{a_{jk} : 1 \le j \le m, 1 \le k \le n\}$ satisfies the bound

$$\Big|\sum_{j,k} a_{jk} x_j y_k\Big| \le M\|x\|_2\|y\|_2, \tag{10.18}$$

then one also has the bound

$$\Big|\sum_{j,k} a_{jk} h_{jk} x_j y_k\Big| \le \alpha\beta M\|x\|_2\|y\|_2 \tag{10.19}$$

provided that the factors h_{jk} have an integral representation of the form

$$h_{jk} = \int_D f_j(x) g_k(x)\, dx \tag{10.20}$$

for which for all j and k one has the bounds

$$\int_D |f_j(x)|^2\, dx \le \alpha^2 \quad \text{and} \quad \int_D |g_k(x)|^2\, dx \le \beta^2. \tag{10.21}$$

Exercise 10.3 (Max Version of Hilbert's Inequality)

Show that for every pair of sequences of real numbers $\{a_n\}$ and $\{b_n\}$ one has

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m b_n}{\max(m,n)} < 4\Big(\sum_{m=1}^{\infty} a_m^2\Big)^{\frac{1}{2}}\Big(\sum_{n=1}^{\infty} b_n^2\Big)^{\frac{1}{2}}, \tag{10.22}$$

and show that 4 may not be replaced by a smaller constant.

Exercise 10.4 (Integral Version)

Prove the integral form of Hilbert's inequality. That is, show that for any $f, g : [0,\infty) \to \mathbb{R}$, one has

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\, dxdy < \pi\Big(\int_0^\infty |f(x)|^2\, dx\Big)^{\frac{1}{2}}\Big(\int_0^\infty |g(y)|^2\, dy\Big)^{\frac{1}{2}}.$$

The discrete Hilbert inequality (10.1) can be used to prove a continuous version, but the strict inequality would be lost in the process. Typically, it is better to mimic the earlier argument rather than to apply the earlier result.

Exercise 10.5 (Homogeneous Kernel Version)

If the function $K : [0,\infty) \times [0,\infty) \to [0,\infty)$ has the homogeneity property $K(\lambda x, \lambda y) = \lambda^{-1} K(x,y)$ for all $\lambda > 0$, then for any pair of functions $f, g : [0,\infty) \to \mathbb{R}$, one has

$$\int_0^\infty \int_0^\infty K(x,y) f(x) g(y)\, dx dy$$
$$< C\biggl(\int_0^\infty |f(x)|^2\, dx \biggr)^{\frac{1}{2}} \biggl(\int_0^\infty |g(y)|^2\, dy \biggr)^{\frac{1}{2}},$$

where the constant C is given by common value of the integrals

$$\int_0^\infty K(1,y) \frac{1}{\sqrt{y}}\, dy = \int_0^\infty K(y,1) \frac{1}{\sqrt{y}}\, dy = \int_1^\infty \frac{K(1,y) + K(y,1)}{\sqrt{y}}\, dy.$$

Exercise 10.6 (The Method of "Parameterized Parameters")

For any positive weights w_k, $k = 1, 2, \ldots, n$, Cauchy's inequality can be restated as a bound on the square of a general sum,

$$(a_1 + a_2 + \cdots + a_n)^2 \le \biggl\{ \sum_{k=1}^n \frac{1}{w_k} \biggr\} \biggl\{ \sum_{k=1}^n a_k^2 w_k \biggr\}, \tag{10.23}$$

and given such a bound it is sometimes useful to note the values w_k, $k = 1, 2, \ldots, n$, can be regarded as *free parameters*. The natural question then becomes, "What can be done with this freedom?" Oddly enough, one may then benefit from introducing yet another real parameter t so that we can write each weight w_k as $w_k(t)$. This purely psychological step hopes to simplify our search for a wise choice of the w_k by refocusing our attention on desirable properties of the *functions* $w_k(t)$, $k = 1, 2, \ldots, n$.

Here we want to squeeze information out of the bound (10.23), and one concrete idea is to look for choices where (1) the first factor of the product (10.23) is bounded uniformly in t and where (2) one can calculate the minimum value over all t of the second factor. These may seem like tall orders, but they can be filled and the next three steps show how this plan leads to some marvelous inferences.

(a) Show that if one takes $w_k(t) = t + k^2/t$ for $k = 1, 2, \ldots, n$ then the first factor of the inequality (10.23) is bounded by $\pi/2$ for all $t \ge 0$ and all $n = 1, 2, \ldots$.

(b) Show that for this choice we also have the identity

$$\min_{t : t \ge 0} \biggl\{ \sum_{k=1}^n a_k^2 w_k(t) \biggr\} = 2 \biggl\{ \sum_{k=1}^n a_k^2 \biggr\}^{\frac{1}{2}} \biggl\{ \sum_{k=1}^n k^2 a_k^2 \biggr\}^{\frac{1}{2}}.$$

(c) Combine the preceding observations to conclude that

$$\left\{\sum_{k=1}^{n} a_k\right\}^4 \leq \pi^2 \left\{\sum_{k=1}^{n} a_k^2\right\}\left\{\sum_{k=1}^{n} k^2 a_k^2\right\}. \tag{10.24}$$

This curious bound is known as Carlson's inequality, and it has been known since 1934. Despite several almost arbitrary steps on the path to the inequality (10.24), the value π^2 cannot be replaced by a smaller one, as one can prove by the stress testing method (page 159), though not without thought.

Exercise 10.7 (Hilbert's Inequality via the Toeplitz Method)

Show that the elementary integral

$$\frac{1}{2\pi}\int_0^{2\pi} (t-\pi)e^{int}dt = \frac{1}{i\,n},$$

for $n \neq 0$, implies that for real a_k, b_k, $1 \leq k \leq N$ one has the integral representation

$$I = \frac{1}{2\pi}\int_0^{2\pi} (t-\pi)\sum_{k=1}^{N} a_k\, e^{ikt}\sum_{k=1}^{N} b_k\, e^{ikt}dt = \sum_{m=1}^{N}\sum_{n=1}^{N}\frac{a_m\, b_n}{m+n},$$

then show that this representation and Schwarz's inequality yield a quick and easy proof of Hilbert's inequality.

Exercise 10.8 (Functional Equation for the Gamma Function)

Recall that the gamma function is defined by the integral

$$\Gamma(\lambda) = \int_0^\infty x^{\lambda-1}e^{-x}\,dx,$$

and use an integral representation for $1/(1+y)$ to show that

$$\int_0^\infty \frac{1}{(1+y)}\frac{1}{y^{2\lambda}}\,dy = \Gamma(2\lambda)\Gamma(1-2\lambda) \qquad \text{for } 0 < \lambda < 1/2. \tag{10.25}$$

As a consequence, one finds that the evaluation of the integral (10.8) yields the famous functional equation for the Gamma function,

$$\Gamma(2\lambda)\Gamma(1-2\lambda) = \frac{\pi}{\sin 2\pi\lambda}.$$