A HARDY-STYLE CONVOLUTION INEQUALITY

ABSTRACT. Inequalities involving convolutions are somewhat under represented in The Cauchy-Schwarz Master Class, and, if there is a second edition, Young's inequality and other convolution problems like the one given below are serious candidates for a chapter of their own.

PROBLEM. Show that for nonnegative a_k and b_k one has

$$\sum_{k=0}^{\infty} \frac{c_k^2}{k+1} \leq \sum_{k=0}^{\infty} a_k^2 \sum_{k=0}^{\infty} b_k^2, \tag{1}$$

where for $0 \leq k < \infty$ we have set

$$c_k = \sum_{j=0}^{k} a_j b_{k-j}.$$

REMARKS.

This is indeed a "Cauchy-Schwarz" problem. It is taken from Ransford (1995), p. 48, where it is applied to prove an inequality for analytic functions, specifically

$$\frac{1}{\pi r^2} \int_{\Delta(0,r)} |f(z)g(z)|^2 dA \leq \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \frac{1}{2\pi} \int_0^{2\pi} |g(re^{i\theta})|^2 d\theta$$

where $\Delta(0, r)$ is the disk of radius r and $dA = r dr d\theta$ is area measure.

REFERENCES

[1] Ransford, T., *Potential Theory in the Complex Plane*, Cambridge University Press, Cambridge UK, 1995.

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