

Book Review

Stochastic Calculus and Financial Applications by J. MICHAEL STEELE.
New York: Springer-Verlag, 2001. Pp. v + 300.

The vast majority of continuous time financial modelling is based on uncertainty driven by Brownian motion. Although pure jump models, as well as jump diffusion models, continue to be developed and implemented in theory and practice, the most popular option pricing models (including fixed income derivative models) exploit the tractability and completeness of diffusion models (i.e. price and portfolio wealth processes being stochastic integrals with respect to Brownian motion). For the student or practitioner wanting to gain a deeper understanding of the mathematics underlying these models, there are several excellent advanced texts on stochastic integration such as Karatzas and Shreve (1991) and Protter (1990); however, these texts have a drawback. By achieving a high degree of generality (e.g. using more general semimartingale integrators), these texts are often unable to give the reader a foundation to appreciate this greater generality. The fact that most of the major probabilistic results used in finance, e.g. Itô's Lemma, Girsanov's Theorem, and the Martingale Representation Theorem, are generalizable beyond Brownian motion uncertainty is not of much use to someone who has trouble understanding these results in a diffusion setting. Just as a strong familiarity with finite dimensional Euclidean geometry is crucial to appreciating the beauty of the more general theory of Hilbert spaces, so it goes with Brownian motion and stochastic integration.

J. Michael Steele has written a book that is a marvelous first step for the person wanting a rigorous development of stochastic calculus, as well as its application to derivative pricing. By focusing solely on Brownian motion, the reader is able to gradually develop an intuition and feel for how to go about solving problems as well as deriving results. By not having to get weighed down in some of the more technical tools required for greater generality, Steele can take the time to describe motivation, take natural detours (that often lead to pitfalls and dead ends), as well as give some historical perspective. In addition, the author is able to convey his passion for this material.

The book can be broken down into four sections: preliminaries (discrete time results and Brownian motion results), Itô analysis (stochastic integration, Itô's lemma, and stochastic differential equations), martingale results (representation theorems, Girsanov theory, and Feynman-Kac), and financial applications. When I first flipped through the pages, I was struck by the relatively large proportion of text (as opposed to mathematics). I initially thought that this implied a book that sacrificed rigor, referring the reader to other texts for proofs. These initial impressions were completely mistaken. This book is remarkably self contained; Steele contains proofs for almost

every result. For the interested reader, I will now give specific details on the book's coverage.

The first two chapters begin with discrete time results. Chapter 1 examines simple random walks, deriving many results (such as expected hitting times and ruin probabilities) using minimal machinery. This leads naturally to Chapter 2's analysis concerning discrete time martingale results (convergence theorems and Doob's inequalities); by returning to some of the previous chapter's problems, Steele is able to demonstrate the power and elegance of the martingale approach. He also introduces stopping times, as well as several martingales whose continuous time counterparts will be introduced later on.

Chapters 3, 4, and 5 lay the ground work for the continuous time analysis. The construction of Brownian motion from wavelets (using normalized Haar and Schauder functions) and scaling and inversion results are presented in Chapter 3. Revisiting the themes of the discrete time chapters, Chapter 4 covers the continuous time version of Chapter 2's martingale results. This requires uniform integrability and formal conditional expectation to be introduced as well as Doob's stopping time theorem. Ruin probabilities as well as important martingales, $B_t^2 - t$ and $\exp(\alpha B_t - \alpha^2 t/2)$, are presented, as well as derivations of ruin probabilities and the distribution of hitting times of Brownian motion. Smoothness properties, reflection and invariance principles, as well as another derivation of the distribution of a hitting time are done in a fairly technical Chapter 5. This chapter concludes with embedding results that informs the reader that the similarity between the discrete time case and the continuous time case is no coincidence.

Having set the stage for Itô calculus, the next two chapters (6 and 7) dive into Itô integration. The approach is pretty standard; start with a small class of integrands and work your way up to the largest class. Steele motivates the final extension to integrands which are square integrable with respect to time (a.s.) by noting that any definition of a stochastic integral should be able to cover the case of integrands that are continuous functions of Brownian motion. After developing this extension, he then applies it to show how stochastic integration with such continuous function integrands can be looked at as a limit (in probability) of Riemann integration. Steele's consistent focus on developing intuition pays off when describing the motivation for and benefit of using localization arguments, which are often hard to appreciate. He concludes Chapter 7 with a proof of how a local martingale remains a local martingale under a time change.

Chapter 8, covering Itô's lemma (which Steele terms Itô's formula), may be the most important chapter for the finance reader. Three versions of the lemma are covered (a scalar function without and with time dependence and a vector extension). In keeping with the author's philosophy, the vector extension's cumbersome proof is "safely omitted". The reader who has used stochastic calculus in practice will find comfort in how many of tricks, such as "box calculus" (e.g. $dB^2 = dt$) and differential notation, can be rigorously justified or interpreted. This chapter also covers recurrence and transience

for two and three (respectively) dimensional Brownian motion, as well as quadratic variation of an Itô process. Stochastic differential equations (SDE) are covered in Chapter 9. Before proving uniqueness and existence, the solutions to geometric Brownian motion and Ornstein-Uhlenbeck (O-U) processes are developed. There is a nice application of the latter to solve a Brownian bridge SDE.

A financial application is finally introduced in Chapter 10's development of Black-Scholes where the stock follows geometric Brownian motion. The replication argument, i.e. finding a self financing stock-bond strategy with a specified payoff, leads to the Black-Scholes partial differential equation (PDE) and the Black-Scholes formula. The method of solving the PDE is put off until the next chapter. What is nice to see is Steele's discussion of the original Black and Scholes (1973) paper. Using their actual formulas (with a change in notation to maintain consistency), Steele takes us through their two strikingly different original derivations. The first is the riskless hedge argument which relies on a riskless stock and call portfolio to arrive at the PDE. I was surprised that Steele did not point out that this riskless hedge strategy is not self financing; although this does not make the resulting PDE any less valid, it is a nice application of Itô's lemma. The second derivation of the PDE relied on a CAPM type argument (where expected returns are linearly related to an asset's Beta). Steele points out that both derivations have their problems (mainly due to imprecision of definitions); however, he stresses that this is the nature of applied mathematics. The original Black-Scholes was more of a motivation of the PDE as a plausible model of option prices rather than a pure theoretical exercise.

Having derived the Black Scholes PDE, Chapter 11 discusses the solutions of diffusion PDEs. This chapter requires a slightly higher level of mathematical background from the reader (such as Fourier transforms, harmonic functions, and some Topology/Real Analysis). After developing existence and uniqueness theorems, the Black-Scholes PDE is solved and is shown to be unique (with a realistic growth rate restriction).

Keeping in step with the historical development of option pricing theory, Steele moves on to develop martingale/Girsanov tools in Chapters 12 and 13. Chapter 12 presents a series of representation theorems; the one of most importance to finance being the martingale representation theorem which proves a martingale can be uniquely represented as a stochastic integral (whose integrand determines the replication strategy). Steele also covers Levy's representation theorem which shows how stochastic integrals can be looked at as Brownian motion under a time change; in describing this connection, one of the author's more memorable paragraphs:

Obviously, this story has more variations than could be imagined even by Sheherazade—a thousand and one nights would never suffice to tell all the tales that Levy's Theorem reads from the marriage of each continuous martingale and each theorem for Brownian motion.

For Chapter 13, covering Girsanov theory, Steele starts with two concrete sections that only require routine manipulation of the normal density. The first section describes importance sampling, a method for estimating (by simulation) tail events by shifting the mean of a normal distribution. The second section looks at how probabilistic descriptions of Brownian motion with drift can be recast as a standard Brownian motion exercise. This allows for Steele to directly calculate the hitting time of a sloping line. Having given the reader a flavor for Girsanov theory, Steele must (finally) introduce the more abstract state space the probability measures are defined on, the set of continuous functions, $C[0, T]$. The result of the second section is then developed more fully, showing how the measure on $C[0, T]$ implied by Brownian motion with drift can be constructed by the measure defined by Brownian motion without drift. Steele then proceeds with the general Girsanov theory; a Brownian motion with a general drift (that satisfies the Novikov condition) can become a standard Brownian motion (martingale) under a new measure. This chapter was very well done; however, I found a feature of this chapter a potential cause of confusion. The first half of the chapter, develops the Girsanov theory by transforming the standard Brownian (driftless) measure to the measure implied by Brownian motion with constant drift; whereas, the general Girsanov result in the second half of the chapter reverses this. It is no coincidence that my copy of this text had a rather glaring typo (which I am told has been corrected in the second printing) that reinforced this confusion.

With the two tools (martingale representation & Girsanov theory) in place, Steele returns to finance in Chapter 14 with a general contingent claim valuation model, where we have general drift and volatility of the (non dividend paying) underlying asset. This is an excellent and rather complete summary of the martingale approach to derivative pricing, as in Harrison and Pliska (1981). Showing how one can construct a replicating strategy when there is a martingale measure, Steele then derives the existence and uniqueness results for such a measure. These results are then used to re-examine the Black-Scholes model and derive the general result of the preclusion of early exercise of convex (path independent) claims. Steele concludes the chapter with some of the subtleties concerning self financing strategies; in order to rule out potentially troublesome strategies that would make derivative prices indeterminate, we restrict strategies to be admissible. Steele then reexamines completeness conditions under this restriction.

Steele chooses to cover Feynman-Kac in the final chapter as it connects the martingale approach with the PDE approach. Besides proving the Feynman-Kac theorem, which represents solutions to a PDE as an expectation under Brownian motion, Steele develops Levy's arcsin law (which calculates the distribution of the proportion of time Brownian motion is positive) and develops a more general (Markov) Black-Scholes model.

There are two appendices; the first containing some mathematical tools concerning basic measure theory and Hilbert space results. The second appendix, Comments and Credits, is a chapter by chapter summary of refer-

ences; this was very complete and deferential. I should add that each chapter concludes with a variety of exercises, many (e.g. the general solution to O-U SDEs) very relevant to finance.

As one can infer, I find little wrong with this book. The two appendices present a natural segue into two minor criticisms. The first has to do with the prerequisites Steele requires from the reader. To quote from the Preface,

The Wharton School course that forms the basis for this book is designed for energetic students who have had some experience with probability and statistics but have not had advanced courses in stochastic processes.

I would be surprised (and humbled) if a student who has not also been exposed to measure theory and real analysis would be able to tackle much of this book (with the exception of the first chapter). Maybe the word “energetic” in the previous quote should be emphasized. In addition, I would suspect the finance chapters would be mysterious for those with no previous exposure to derivative pricing. Secondly, it would be nice to have a final chapter giving the reader an idea on how the main results are generalizable, pointing out the limitations of the book.

I'd like to conclude this review with a comparison of a book I find to be similar in spirit; in fact, it would be a wonderful book to tackle before Steele's. As a graduate student in financial economics, wanting to learn relevant measure theoretic probability, one of my favorite books was David Williams' *Probability with Martingales* (1991). Not only was it slimmer and more enjoyable to read than the standard and more complete texts, e.g. Billingsley (1986), but it also maintained a high degree of rigor. Of course, this implied a sacrifice of coverage; however, the central theme of the book (discrete time martingales) happened to be an important element for financial applications.

I believe the same can be said for J. Michael Steele's book. If a reader wants to tackle stochastic integration, they could start by investing their time and dollars in more encompassing and advanced texts that they could use as references for their future work; however, the required terseness and “Lemma, Exercise, Proposition” style may quickly turn off the uninitiated. In addition, without discussions of motivation and intuition, the reader is often mechanically following along rather than developing a way of thinking about general problems. This might work fine for a reader who gets this more general guidance from a professor in a classroom. But for the practitioner or researcher who is not able to be in a classroom, Steele's book will give that less tangible benefit. Although the text would work wonderfully as a course textbook, I believe this book differentiates itself as the clear choice for an outside the classroom reader wanting a remarkably complete development of stochastic integration and stochastic differential equations under Brownian motion. I would expect this book will become very popular for Ph.D. students in finance who are comfortable with real analysis and are

wanting a rigorous, well written, and enjoyable treatment of stochastic calculus.

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