QUICKEST ONLINE SELECTION OF AN INCREASING SUBSEQUENCE OF SPECIFIED SIZE

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Abstract. Given a sequence of independent random variables with a common continuous distribution, we consider the online decision problem where one seeks to *minimize the expected value of the time* that is needed to complete the selection of a monotone increasing subsequence of a prespecified length n. This problem is dual to the online decision problems that have been considered earlier, and this dual problem has some notable advantages. In particular, the recursions and equations of optimality lead with relative ease to asymptotic formulas for mean and variance of the minimal selection time.

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1. Increasing Subsequences and Time Focused Selection

If $X_1, X_2, \ldots$ is a sequence of independent random variables with a common continuous distribution F, then

$$L_n = \max\{k : X_{i_1} < X_{i_2} < \cdots < X_{i_k}, \text{ where } 1 \le i_1 < i_2 < \cdots < i_k \le n\}$$

represents the length of the longest monotone increasing subsequence in the sample $\{X_1, X_2, \ldots, X_n\}$. This random variable was considered by Ulam (1961) in the early days of the Monte Carlo method, but the probability theory of L_n was first engaged in earnest by Hammersley (1972) who used a clever subadditive argument to show that $\mathbb{E}[L_n] \sim c\sqrt{n}$ as $n \to \infty$ for a constant $c \in (\pi/2, e)$. Not long afterwards, Veršik and Kerov (1977) and Logan and Shepp (1977) proved that $c = 2$. Much later, through a remarkable sequence of developments culminating with Baik, Deift and Johansson (1999), it was found that $n^{-1/6}(L_n - 2\sqrt{n})$ converges in distribution to the Tracy-Widom law, a new universal law introduced a few years earlier in Tracy and Widom (1994). The review of Aldous and Diaconis (1999) and the monograph of Romik (2015) draw connections between the increasing subsequence problem and topics as diverse as card sorting, triangulation of Riemann surfaces, and the theory

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of partitions. Still more recent variations on the monotone subsequence problem have been analyzed in Bhatnagar and Peled (2014) and Kiwi and Soto (2015).

Here we consider an online decision problem where the decision maker's task is to select *as quickly as possible* an increasing subsequence of length n. More precisely, at time i, when the decision maker is first presented with X_i, a decision must be made either to accept X_i a member of the selected subsequence or else to reject X_i forever. The decision at time i is assumed to be a deterministic function of the observations $\{X_1, X_2, \ldots, X_i\}$, so the times $1 \le \tau_1 < \tau_2 < \cdots < \tau_n$ of affirmative selections give us a strictly increasing sequence stopping times that are adapted to the sequence of σ-fields $\mathcal{F}_i = \sigma\{X_1, X_2, \ldots, X_i\}$, $1 \le i < \infty$.

Here the quantity of most interest is

$$\beta(n) := \min_{\pi} \mathbb{E}[\tau_n] \tag{1}$$

where the minimum is over all sequences $\pi = (\tau_1, \tau_2, \ldots, \tau_n)$ of stopping times such that

$$1 \le \tau_1 < \tau_2 < \cdots < \tau_n \quad \text{and} \quad X_{\tau_1} < X_{\tau_2} < \cdots < X_{\tau_n}.$$

Such a sequence π will be called a *selection policy*, and the set of all such selection policies with $\mathbb{E}[\tau_n] < \infty$ will be denoted by $\Pi(n)$.

It is useful to note that the value of $\beta(n)$ is not changed if we replace each X_i with $F^{-1}(X_i)$, so we may as well assume from the beginning that the X_i's are all uniformly distributed on $[0, 1]$. Our main results concern the behavior of $\beta(n)$ for each $n \ge 1$ and the structure of the policy that attains the minimum (1).

Theorem 1. *The function*

$$n \mapsto \beta(n) = \min_{\pi \in \Pi(n)} \mathbb{E}[\tau_n]$$

is convex, it has initial value $\beta(1) = 1$ *and for all* $n \ge 2$ *it satisfies the bounds*

$$\frac{1}{2}n^2 \le \beta(n) \le \frac{1}{2}n^2 + n \log n. \tag{2}$$

One can add some precision to this result by focusing on the subclass of threshold policies. These are the policies $\pi = (\tau_1, \tau_2, \ldots, \tau_n) \in \Pi(n)$ that are determined by a sequence of real values $\{t_i \in [0, 1] : 1 \le i \le n\}$ and the corresponding recursion

$$\tau_{k+1} = \min\{i > \tau_k : X_i \in [X_{\tau_k},\, X_{\tau_k} + t_{n-k}(1 - X_{\tau_k})]\}, \quad 0 \le k < n, \tag{3}$$

where the recursion begins with $\tau_0 = 0$ and $X_0 = 0$. Here one can think of t_{n-k} as the "threshold parameter" that specifies the maximum fraction that one would be willing to spend from a "residual budget" $(1 - X_{\tau_k})$ to accept a value that arrives after the time τ_k when the k'th selection was made.

Theorem 2. *There is a unique threshold policy* $\pi^* = (\tau_1^*, \tau_2^*, \ldots, \tau_n^*) \in \Pi(n)$ *for which one has*

$$\beta(n) = \min_{\pi \in \Pi(n)} \mathbb{E}[\tau_n] = \mathbb{E}[\tau_n^*], \tag{4}$$

and for this optimal policy π^* *one has for all* $\alpha > 2$ *that*

$$\mathrm{Var}[\tau_n^*] = \frac{1}{3}n^3 + O(n^2 \log^\alpha n) \quad \text{as } n \to \infty. \tag{5}$$

In the next section, we prove the existence and uniqueness of an optimal threshold policy, and in Section 3 we complete the proof of Theorem 1 after deriving some recursions that permit the exact computation of the optimal threshold values. Section 4 deals with the asymptotics of the variance and completes the proof of Theorem 2. In Sections 5 and 6, we use Theorems 1 and 2 and show how they can be used to recover some previously known results for the traditional online size-focused increasing subsequence problem. Finally, in Section 7 we comment briefly on both alternative methods and underscore some open problems.

2. Threshold Policies: Existence and Optimality

A beneficial feature of the time-focused monotone selection problem is that there is a natural similarity relationship between the problems of size n and size $n-1$. This "scaled regeneration" leads one to a useful recursion for $\beta(n)$.

Lemma 1 (Variational Beta Recursion)**.** *For all $n = 1, 2, \ldots$ we have*

$$\beta(n) = \inf_{\tau} \mathbb{E}\left[\tau + \frac{1}{1 - X_\tau}\beta(n-1)\right], \tag{6}$$

where the minimum is over all stopping times τ and where we initialize the recursion by setting $\beta(0) = 0$.

Proof. To argue by induction, we first note that $\beta(1) = 1$, so one can confirm (6) simply by taking $\tau = 1$. Now take $n \geq 2$ and consider any selection policy $\pi = (\tau_1, \tau_2, \ldots, \tau_n)$. If we set $\pi' = (\tau_2 - \tau_1, \tau_3 - \tau_1, \ldots, \tau_n - \tau_1)$, then one can view π' as a selection policy for the sequence $(X'_1, X'_2, \ldots) = (X_{1+\tau_1}, X_{2+\tau_1}, \ldots)$ where one can only make selections from those values that fall in the interval $[X_{\tau_1}, 1]$. If we condition on τ_1 and X_{τ_1}, then the definition of $\beta(n-1)$ gives us the inequality

$$\tau_1 + \frac{\beta(n-1)}{(1 - X_{\tau_1})} \leq \tau_1 + \mathbb{E}[\tau_n - \tau_1 \,|\, \tau_1, X_{\tau_1}],$$

so, if we take the total expectation then the definition of $\beta(n)$ gives us

$$\beta(n) \leq \mathbb{E}[\tau_1 + \frac{\beta(n-1)}{(1 - X_{\tau_1})}] \leq \mathbb{E}[\tau_n]. \tag{7}$$

Now, we take the infimum in (7) over all $\pi = (\tau_1, \tau_2, \ldots, \tau_n) \equiv (\tau, \tau_2, \ldots, \tau_n)$ in $\Pi(n)$. After this, we use the definition of $\beta(n) = \min_\pi \mathbb{E}[\tau_n]$ one more time to get a sandwich inequality that proves the identity (6). □

The recursion (6) has several uses. In particular, it helps one to show that there is a unique threshold policy that achieves the minimal expectation $\beta(n)$.

Lemma 2 (Existence and Uniqueness of an Optimal Threshold Policy)**.** *There are constants $0 \leq t_i \leq 1$, $1 \leq i \leq n$, such that the threshold policy $\pi^* \in \Pi(n)$ defined by (3) is the unique optimal policy. That is, for $\pi^* = (\tau_1^*, \tau_2^*, \ldots, \tau_n^*)$ one has*

$$\beta(n) = \min_{\pi \in \Pi(n)} \mathbb{E}[\tau_n] = \mathbb{E}[\tau_n^*], \tag{8}$$

and π^* is the only *policy in $\Pi(n)$ that achieves this minimum.*

Proof. The proof again proceeds by induction. The case $n = 1$ is trivial since the only optimal policy is to take any element which presented; this is the threshold policy with $t_1 = 1$ and $\beta(1) = 1$.

For the moment, we consider an arbitrary policy $\pi = (\tau_1, \tau_2, \ldots, \tau_n) \in \Pi(n)$. We have $1 \leq \mathbb{E}[\tau_1] < \infty$, and we introduce a parameter t by setting $t = (\mathbb{E}[\tau_1])^{-1}$. Next, we define a new, threshold stopping time τ_1^* by setting

$$\tau_1^* = \min\{i : X_i < t\},$$

and we note that this construction gives us $\mathbb{E}[\tau_1^*] = \mathbb{E}[\tau_1] = 1/t$. For $s \in [0, t]$, we also have the trivial inequality

$$\mathbb{1}(X_{\tau_1} < s) \leq \sum_{i=1}^{\tau_1} \mathbb{1}(X_i < s),$$

so by Wald's equation we also have

$$\mathbb{P}(X_{\tau_1} < s) \leq \mathbb{E}[\sum_{i=1}^{\tau_1} \mathbb{1}(X_i < s)] = s\mathbb{E}[\tau_1] = s/t. \tag{9}$$

The definition of τ_1^* implies that $X_{\tau_1^*}$ is uniformly distributed on $[0, t]$, so we further have $\mathbb{P}(X_{\tau_1^*} < s) = \min\{1, s/t\}$, so comparison with (9) gives us the domination relation

$$\mathbb{P}(X_{\tau_1} < s) \leq \mathbb{P}(X_{\tau_1^*} < s) \quad \text{for all } 0 \leq s \leq 1. \tag{10}$$

From (10) and the monotonicity of $x \mapsto (1-x)^{-1}$, we have by integration that

$$\mathbb{E}[\frac{\beta(n-1)}{1 - X_{\tau_1^*}})] \leq \mathbb{E}[\frac{\beta(n-1)}{1 - X_{\tau_1}}]; \tag{11}$$

moreover, one has a strict inequality in (10) and (11) unless $\tau_1^* = \tau_1$ with probability one.

If we now add $\mathbb{E}[\tau_1^*] = \mathbb{E}[\tau_1]$ to the corresponding sides of (11) and take the infimum over all τ_1, then the beta recursion (6) gives us

$$\mathbb{E}[\tau_1^*] + \mathbb{E}\left[\frac{\beta(n-1)}{1 - X_{\tau_1^*}})\right] \leq \inf_{\tau_1} \left\{\mathbb{E}[\tau_1] + \mathbb{E}[\frac{\beta(n-1)}{1 - X_{\tau_1}}]\right\} = \beta(n).$$

In other words, the first selection of an optimal policy is given by uniquely by a threshold rule.

To see that all subsequent selections must be made by threshold rules, we just need to note that given the time τ_1 and value $X_{\tau_1} = x$ of the first selection, one is left with a selection problem of size $n-1$ from the smaller set $\{X_i : i > \tau_1 \text{ and } X_i > x\}$. The induction hypothesis applies to this problem of size $n-1$, so we conclude that there is a unique threshold policy $(\tau_2^*, \tau_3^*, \ldots, \tau_n^*)$ that is optimal for these selections. Taken as a whole, we have a unique threshold policy $(\tau_1^*, \tau_2^*, \ldots, \tau_n^*) \in \Pi(n)$ for the problem of selecting an increasing subsequence of size n in minimal time. □

Lemma 2 completes the proof of the first assertion (4) of Theorem 2. After we develop a little more information on the behavior of the mean, we will return to the proof of the second assertion (5) of Theorem 2.

3. Lower and Upper Bounds for the Mean

The recursion (6) for $\beta(n)$ is informative, but to determine the asymptotic behavior of $\beta(n)$, we need more concrete and more structured recursions. The key relations are summarized in the next lemma.

Lemma 3 (Recursions for $\beta(n)$ and the Optimal Thresholds)**.** *For each $x \geq 1$ and $t \in (0,1)$ we let*

$$g(x,t) = \frac{1}{t} + \frac{x}{t} \log\left(\frac{1}{1-t}\right), \quad G(x) = \min_{0<t<1} g(x,t), \quad \text{and} \tag{12}$$

$$H(x) = \arg\min_{0<t<1} g(x,t). \tag{13}$$

We then have $\beta(1) = 1$, and we have the recursion

$$\beta(n+1) = G(\beta(n)) \quad \text{for all } n \geq 1. \tag{14}$$

Moreover, if the deterministic sequence $t_1, t_2, \ldots$ is defined by the recursion

$$t_1 = 1 \quad \text{and} \quad t_{n+1} = H(\beta(n)) \quad \text{for all } n \geq 1, \tag{15}$$

then the minimum in the defining equation (1) *for $\beta(n)$ is uniquely achieved by the sequence of stopping times given by the threshold recursion* (3)*.*

Proof. An optimal first selection time has the form $\tau_1 = \min\{i : X_i < t\}$, so we can rewrite the recursion (6) as

$$\begin{aligned}\beta(n) &= \min_{0<t<1} \left\{\frac{1}{t} + \mathbb{E}[\frac{\beta(n-1)}{1-X_{\tau_1}}]\right\} = \min_{0<t<1} \left\{\frac{1}{t} + \frac{\beta(n-1)}{t} \int_0^t \frac{1}{1-s}\, ds\right\} \\ &= \min_{0<t<1} g(t, \beta(n-1)) \equiv G(\beta(n-1)).\end{aligned} \tag{16}$$

The selection rule for the first element is given by $\tau_1 = \min\{i : X_i < t_n\}$ so by (16) and the definitions of g and H we have $t_n = H(\beta(n-1))$. □

Lemma 3 already gives us enough to prove the first assertion of Theorem 1 which states that the map $n \mapsto \beta(n)$ is convex.

Lemma 4. *The map $n \mapsto \Delta(n) := \beta(n+1) - \beta(n)$ is an increasing function.*

Proof. One can give a variational characterization of Δ that makes this evident. First, by the defining relations (12) and the recursion (14) we have

$$\begin{aligned}\beta(n+1) - \beta(n) &= G(\beta(n)) - \beta(n) \\ &= \min_{0<t<1} \left\{\frac{1}{t} + \beta(n)\left[\frac{1}{t}\log\left(\frac{1}{1-t}\right) - 1\right]\right\},\end{aligned}$$

so if we set

$$\widehat{g}(x,t) = \frac{1}{t} + x\left[\frac{1}{t}\log\left(\frac{1}{1-t}\right) - 1\right],$$

then we have

$$\Delta(n) = \beta(n+1) - \beta(n) = \min_{0<t<1} \widehat{g}(\beta(n), t). \tag{17}$$

Now, for $0 \leq x \leq y$ and $t \in (0,1)$ we then have

$$\widehat{g}(x,t) - \widehat{g}(y,t) = (x-y)\left[\frac{1}{t}\log\left(\frac{1}{1-t}\right) - 1\right] = (x-y)\sum_{k=2}^{\infty} \frac{1}{k} t^{k-1} \leq 0;$$

so from the monotonicity $\beta(n) \leq \beta(n+1)$, we get

$$\widehat{g}(\beta(n), t) \leq \widehat{g}(\beta(n+1), t) \qquad \text{for all } 0 < t < 1.$$

When we minimize over $t \in (0,1)$, we see that (17) gives us $\Delta(n) \leq \Delta(n+1)$. □

We next show that the two definitions in (12) can be used to give an *a priori* lower bound on G. An induction argument using the recursion (14) can then be used to obtain the lower half of (2).

Lemma 5 (Lower Bounding G Recursion)**.** *For the function $x \mapsto G(x)$ defined by* (12)*, we have*

$$\frac{1}{2}(x+1)^2 \le G\left(\frac{x^2}{2}\right) \qquad \text{for all } x \ge 1. \tag{18}$$

Proof. To prove (18), we first note that by (12) it suffices to show that one has

$$\delta(x,t) = (x+1)^2 t - 2 - x^2 \log\left(\frac{1}{1-t}\right) \le 0 \tag{19}$$

for all $x \ge 1$ and $t \in (0,1)$. For $x \ge 1$ the map $t \mapsto \delta(x,t)$ is twice-continuous differentiable and concave in t. Hence there is a unique value $t^* \in (0,1)$ such that $t^* = \text{argmax}_{0<t<1}\delta(x,t)$, and such that $t^* = t^*(x)$ satisfies the first order condition

$$(x+1)^2 - (1-t^*)^{-1}x^2 = 0.$$

Solving this equation gives us

$$t^* = \frac{2x+1}{(x+1)^2}, \quad \text{and} \quad \delta(x,t^*) = -1 + 2x - 2x^2 \log\left(1 + \frac{1}{x}\right),$$

so the familiar bound

$$\frac{1}{x} - \frac{1}{2x^2} \le \log\left(1 + \frac{1}{x}\right) \qquad \text{for } x \ge 1,$$

gives us

$$\delta(x,t) \le \delta(x,t^*) \le -1 + 2x - 2x^2\left(\frac{1}{x} - \frac{1}{2x^2}\right) = 0,$$

and this is just what we needed to complete the proof of (19). □

Now, to argue by induction, we consider the hypothesis that one has

$$\frac{1}{2}n^2 \le \beta(n). \tag{20}$$

This holds for $n = 1$ since $\beta(1) = 1$, and, if it holds for some $n \ge 1$, then by the monotonicity of G we have $G(n^2/2) \le G(\beta(n))$. Now, by (18) and (14) we have

$$\frac{1}{2}(n+1)^2 \le G(n^2/2) \le G(\beta(n)) = \beta(n+1),$$

and this completes our induction step from (20). Finally, to complete the proof of Theorem 1, it only remains to prove the upper half of (2). The argument again depends on an *a priori* bound on G. The proof is brief but delicate.

Lemma 6 (Upper Bounding G Recursion)**.** *For the function $x \mapsto G(x)$ defined by* (12) *one has*

$$G(\frac{1}{2}x^2 + x\log(x)) \le \frac{1}{2}(x+1)^2 + (x+1)\log(x+1) \qquad \text{for all } x \ge 1.$$

Proof. If we set $f(x) := x^2/2 + x\log(x)$, then we need to show that

$$G(f(x)) \le f(x+1).$$

If we take $t' = 2/(x+2)$ then the defining relation (12) for G tells us that

$$G(f(x)) \le g(f(x), t') = \frac{x+2}{2} + \frac{x+2}{2}\log\left(1 + \frac{2}{x}\right) f(x). \tag{21}$$

Next, for any $y \ge 0$ integration over $(0, y)$ of the inequality

$$\frac{1}{u+1} \le \frac{u^2+2u+2}{2(u+1)^2} \quad \text{implies the bound} \quad \log(1+y) \le \frac{y(y+2)}{2(y+1)}.$$

If we now set $y = 2/x$ and substitute this last bound in (21), we obtain

$$\begin{aligned} G(f(x)) &\le \frac{x}{2} + 1 + \left(1 + \frac{1}{x}\right) f(x) \\ &= f(x+1) + \frac{1}{2} + (x+1)\{\log(x) - \log(x+1)\} \le f(x+1), \end{aligned}$$

just as needed to complete the proof of the lemma. □

One can now use Lemma 6 and induction to prove that for all $n \ge 2$ one has

$$\beta(n) \le \frac{n^2}{2} + n\log(n). \tag{22}$$

Since $\beta(2) = G(1) = \min_{0<t<1} g(1,t) < 3.15$ and $2(1+\log(2)) \approx 3.39$, one has (22) for $n = 2$. Now, for $n \ge 2$, the monotonicity of G gives us that

$$\beta(n) \le \frac{1}{2}n^2 + n\log(n) \quad \text{implies} \quad G(\beta(n)) \le G(\frac{1}{2}n^2 + n\log(n)).$$

Finally, by the recursion (14) and Lemma 6 we have

$$\begin{aligned} \beta(n+1) = G(\beta(n)) &\le G(\frac{1}{2}n^2 + n\log(n)) \\ &\le \frac{1}{2}(n+1)^2 + (n+1)\log(n+1). \end{aligned}$$

This completes the induction step and establishes (22) for all $n \ge 2$. This also completes last part of the proof of Theorem 1.

4. Asymptotics for the Variance

To complete the proof of Theorem 2, we only need to prove that one has the asymptotic formula (5) for $\mathrm{Var}[\tau_n^*]$. This will first require an understanding of the size of the threshold t_n, and we can get this from our bounds on $\beta(n)$ once we have an asymptotic formula for H. The next lemma gives us what we need.

Lemma 7. *For $x \mapsto G(x)$ and $x \mapsto H(x)$ defined by* (12) *and* (13)*, we have for $x \to \infty$ that*

$$G(x) = (x^{1/2} + 2^{-1/2})^2(1 + O(1/x)) \quad \textit{and} \tag{23}$$

$$H(x) = (2/x)^{1/2}(1 + O(x^{-1/2})). \tag{24}$$

Proof. For any fixed $x \geq 1$ we have $g(t,x) \to \infty$ when $t \to 0$ or $t \to 1$, so the minimum of $g(t,x)$ is obtained at an interior point $0 < t < 1$. Computing the t-derivative $g_t(t,x)$ gives us

$$g_t(t,x) = -\frac{1}{t^2} - \frac{x}{t^2}\log(\frac{1}{1-t}) + \frac{x}{t(1-t)},$$

so the first order condition $g_t(t,x) = 0$ implies that at the minimum we have the condition

$$\frac{1}{t^2} = -\frac{x}{t^2}\log(\frac{1}{1-t}) + \frac{x}{t(1-t)}.$$

Writing this more informatively as

$$\frac{1}{x} = \log(1-t) + \frac{t}{1-t} = \frac{t^2}{2} + \sum_{i=3}^{\infty} \frac{i-1}{i} t^i, \tag{25}$$

we see the right-hand side is monotone in t, so there is a unique value $t_* = t_*(x)$ that solves (25) for t. The last sum on the right-hand side of (25) tells us that

$$\frac{1}{2}t_*^2 \leq \frac{1}{x} \quad \text{or, equivalently,} \quad t_* \leq \sqrt{\frac{2}{x}},$$

and when we use these bounds in (25) we have

$$\frac{t_*^2}{2} \leq \frac{1}{x} \leq \frac{t_*^2}{2} + \sum_{i=3}^{\infty}(\frac{2}{x})^{i/2} \leq \frac{t_*^2}{2} + O(x^{-3/2}).$$

Solving these inequalities for t_*, we then have by the definition (13) of H that

$$H(x) = t_* = \sqrt{\frac{2}{x}}\big(1 + O(x^{-1/2})\big). \tag{26}$$

Finally, to confirm the approximation (23), we substitute $H(x) = t_*$ into the definition (12) of G and use the asymptotic formula (26) for $H(x)$ to compute

$$\begin{aligned} G(x) = g(x,t_*) &= \frac{1}{t_*} + \frac{x}{t_*}\log\left(\frac{1}{1-t_*}\right) = \frac{1}{t_*}(1 + x\sum_{i=1}^{\infty}\frac{t_*^i}{i}) \\ &= \frac{1}{t_*}(1 + xt_* + \frac{xt_*^2}{2} + O(x{t_*}^3)) = \frac{1}{t_*}(xt_* + 1 + \frac{xt_*^2}{2}) + O(1) \\ &= x + 2\sqrt{\frac{x}{2}} + O(1) = \Big(x^{1/2} + 2^{-1/2}\Big)^2\Big(1 + O(\frac{1}{x})\Big), \end{aligned}$$

and this completes the proof of the lemma. □

The recursion (15) tells us that $t_n = H(\beta(n))$ and the upper and lower bounds (2) of Theorem 1 tell us that $\beta(n) = n^2/2 + O(n\log n)$, so by the asymptotic formula (24) for H we have

$$t_n = \frac{2}{n} + O(n^{-2}\log n). \tag{27}$$

To make good use of this formula we only need two more tools. First, we need to note that random τ_n^* satisfies a naturally distributional identity. This will lead in turn to a recursion from which we can extract the required asymptotic formula for $v(n) = \mathrm{Var}(\tau_n^*)$.

If t_n is the threshold value defined by the recursion (15), we let $\gamma(t_n)$ denote a geometric random variable of parameter $p = t_n$ and we let $U(t_n)$ denote a random variable with the uniform distribution on the interval $[0, t_n]$. Now, if we take the random variables $\gamma(t_n)$, $U(t_n)$, and τ_{n-1} to be independent, then we have the distributional identity,

$$\tau_n^* \stackrel{d}{=} \gamma(t_n) + \frac{\tau_{n-1}^*}{(1 - U(t_n))}, \tag{28}$$

and this leads to a useful recursion for the variance of τ_n^*. To set this up, we first put

$$R(t) = (1 - U(t))^{-1}$$

where $U(t)$ is uniformly distributed on $[0, t]$, and we note

$$\mathbb{E}[R(t)] = -t^{-1} \log(1 - t) = 1 + t/2 + t^2/3 + O(t^3); \tag{29}$$

moreover, since $\mathbb{E}[R^2(t)] = (1 - t)^{-1} = 1 + t + t^2 + O(t^3)$, we also have

$$\mathrm{Var}[R(t)] = (1 - t)^{-1} - t^{-2} \log^2(1 - t) = \frac{t^2}{12} + O(t^3). \tag{30}$$

Lemma 8 (Approximate Variance Recursion)**.** *For the variance $v(n) := \mathrm{Var}(\tau_n^*)$ one has the approximate recursion*

$$v(n) = \Big(1 + \frac{2}{n} + O(\frac{\log n}{n^2})\Big) v(n - 1) + \frac{n^2}{3} + O(n \log n). \tag{31}$$

Proof. By independence of the random variables on the right side of (28), we have

$$v(n) = \mathrm{Var}(\gamma(t_n)) + \mathrm{Var}[R(t_n)\tau_{n-1}^*]. \tag{32}$$

From (27) we have $t_n = 2/n + O(n^{-2} \log n)$, so for the first summand we have

$$\mathrm{Var}(\gamma(t_n)) = \frac{1}{t_n^2} - \frac{1}{t_n} = \frac{n^2}{4} + O(n \log n). \tag{33}$$

To estimate the second summand, we first use the complete variance formula and independence to get

$$\begin{aligned} \mathrm{Var}(R(t_n)\tau_{n-1}^*) &= \mathbb{E}[R^2(t_n)]\mathbb{E}[(\tau_{n-1}^*)^2] - \mathbb{E}[R(t_n)]^2(\mathbb{E}[\tau_{n-1}^*])^2 \\ &= \mathbb{E}[R^2(t_n)]v(n - 1) + \mathrm{Var}[R(t_n)](\mathbb{E}[\tau_{n-1}^*])^2. \end{aligned} \tag{34}$$

Now from (27) and (30) we have

$$\mathrm{Var}[R(t_n)] = \frac{1}{3} n^{-2} + O(n^{-3} \log n),$$

and from (2) we have

$$(\mathbb{E}[\tau_{n-1}^*])^2 = \{(n - 1)^2/2 + O(n \log n)\}^2 = \frac{1}{4} n^4 + O(n^3 \log n),$$

so from (32), (33) and (34) we get

$$v(n) = \{1 + \frac{2}{n} + O(n^{-2} \log n)\} v(n - 1) + \frac{n^2}{3} + O(n \log n),$$

and this completes the proof of (31). ∎

To conclude the proof of Theorem 2, it only remains to show that the approximate recursion (31) implies the asymptotic formula

$$v(n) = \frac{1}{3} n^2(n + 1) + O(n^2 \log^\alpha n) \quad \text{for } \alpha > 2. \tag{35}$$

If we define $r(n)$ by setting $v(n) = 3^{-1}n^2(n+1) + r(n)$, then substitution of $v(n)$ into (31) gives us a recursion for $r(n)$,

$$r(n) = (1 + 2/n + O(n^{-2}\log n))r(n-1) + O(n\log n).$$

We then consider the normalized values $\hat{r}(n) = r(n)/(n^2\log^{\alpha}(n))$, and we note they satisfy the recursion

$$\hat{r}(n) = (1 + O(n^{-2}))\hat{r}(n-1) + O(n^{-1}\log^{\alpha-1} n). \tag{36}$$

This is a recursion of the form $\hat{r}(n+1) = \rho_n\hat{r}(n) + \epsilon_n$, and one finds by induction its solution has the representation

$$\hat{r}(n) = \hat{r}(0)\rho_0\rho_1\cdots\rho_{n-1} + \sum_{k=0}^{n-1}\epsilon_k\rho_{k+1}\cdots\rho_{n-1}.$$

Here, the product of the "evolution factors" ρ_n is convergent and the sum of the "impulse terms" ϵ_n is finite, so the sequence $\hat{r}(n)$ is bounded, and, consequently, (31) gives us (35). This completes the proof of the last part of Theorem 2.

5. Suboptimal Policies and a Blocking Inequality

Several inequalities for $\beta(n)$ can be obtained through the construction of suboptimal policies. The next lemma illustrates this method with an inequality that leads to an alternative proof of (20), the uniform lower bound for $\beta(n)$.

Lemma 9 (Blocking Inequality). *For nonnegative integers n and m one has the inequality*

$$\beta(mn) \le \min\{m^2\beta(n), n^2\beta(m)\}. \tag{37}$$

Proof. First, we fix n and we consider a policy π^* that achieves the minimal expectation $\beta(n)$. The idea is to use π^* to build a suboptimal π' policy for the selection of an increasing subsequence of length mn. We take X_i, $i = 1, 2, \ldots$ to be a sequence of independent random variables with the uniform distribution on $[0,1]$, and we partition $[0,1]$ into the subintervals $I_1 = [0, 1/m)$, $I_2 = [1/m, 2/m)$, ..., $I_m = [(m-1)/m, 1]$. We define π' by three rules:

(i) Beginning with $i = 1$ we say X_i is feasible value if $X_i \in I_1$. If X_i is feasible, we accept X_i if X_i would be accepted by the policy π^* applied to the sequence of feasible values after we rescale those values to be uniform in $[0,1]$. We continue this way until the time τ'_1 when we have selected n values.

(ii) Next, beginning with $i = \tau'_1$, we follow the previous rule except that now we say X_i is feasible value if $X_i \in I_2$. We continue in this way until time $\tau'_1 + \tau'_2$ when n additional increasing values have been selected.

(iii) We repeat this process $m-2$ more times for the successive intervals I_3, I_4,..., I_m.

At time $\tau'_1 + \tau'_2 + \cdots + \tau'_m$, the policy π' will have selected nm increasing values. For each $1 \le j \le m$ we have $\mathbb{E}[\tau'_j] = m\beta(n)$, so by suboptimality of π' we have

$$\beta(mn) \le \mathbb{E}[\tau'_1 + \tau'_2 + \cdots + \tau'_m] = m^2\beta(n).$$

We can interchange the roles of m and n, so the proof of the lemma is complete. $\square$

The blocking inequality (37) implies that even the crude asymptotic relation $\beta(n) = \frac{1}{2}n^2 + o(n^2)$ is strong enough to imply the uniform lower bound $\frac{1}{2}n^2 \le \beta(n)$. Specifically, one simply notes from (37) and $\beta(n) = \frac{1}{2}n^2 + o(n^2)$ that

$$\frac{\beta(mn)}{(mn)^2} \le \frac{\beta(n)}{n^2} \quad \text{and} \quad \lim_{m\to\infty} \frac{\beta(mn)}{(mn)^2} = \frac{1}{2}.$$

This derivation of the uniform bound $\frac{1}{2}n^2 \le \beta(n)$ seems to have almost nothing in common with the proof by induction that was used in the proof of Lemma 6. Still, it does require the bootstrap bound $\beta(n) = \frac{1}{2}n^2 + o(n^2)$, and this does require at least some of the machinery of Lemma 3.

6. Duality and the Size-Focused Selection Problem

In the online *size-focused* selection problem one considers a set of policies $\Pi_s(n)$ that depend on the size n of a sample $\{X_1, X_2, \dots, X_n\}$, and the goal is to make sequential selections in order to maximize the expected size of the selected increasing subsequence. More precisely, a policy $\pi_n \in \Pi_s(n)$ is determined by stopping times τ_i, $i = 1, 2, \dots$ such that $1 \le \tau_1 < \tau_2 < \cdots < \tau_k \le n$ and $X_{\tau_1} \le X_{\tau_2} \le \cdots \le X_{\tau_k}$. The random variable of interest is

$$L_n^o(\pi_n) = \max\{k : X_{\tau_1} < X_{\tau_2} < \cdots < X_{\tau_k} \text{ where } 1 \le \tau_1 < \tau_2 < \cdots < \tau_k \le n\}, \tag{38}$$

and most previous analyses have focused on the asymptotic behavior of

$$\ell(n) := \max_{\pi_n \in \Pi_s(n)} \mathbb{E}[L_n^o(\pi_n)]. \tag{39}$$

For example, Samuels and Steele (1981) found that $\ell(n) \sim \sqrt{2n}$, but now a number of refinements of this are known. Our goal here is to show how some of these refinements follow from the preceding theory.

Uniform Upper Bound for $\ell(n)$ via Duality.

Perhaps the most elegant refinement of $\ell(n) \sim \sqrt{2n}$ is the following uniform upper bound that follows from the related analysis of Bruss and Robertson (1991) and Gnedin (1999).

Proposition 1 (Uniform Upper Bound)**.** *For all $n \ge 1$, one has*

$$\ell(n) \le \sqrt{2n}. \tag{40}$$

This proposition is now well understood, but it is instructive to see how it can be derived from $\beta(n) = (1/2)n^2 + O(n \log n)$. The basic idea is to exploit duality with a suboptimality argument like the one used in Section 5, but here a bit more work is required.

We fix n, and, for a much larger integer k, we set

$$N_k = \lfloor (k - 2k^{2/3})\ell(n) \rfloor \quad \text{and} \quad r_k = \lfloor k - k^{2/3} \rfloor. \tag{41}$$

The idea of the proof is to give an algorithm that is guaranteed to select from $\{X_1, X_2, \dots\}$ an increasing subsequence of length N_k. If T_k is the number of the elements that the algorithm inspects before returning the increasing subsequence, then by the definition of $\beta(\cdot)$ we have $\beta(N_k) \le \mathbb{E}[T_k]$; one then argues that (40) follows from this relation.

We now consider $[0,1]$ and for $1 \le i \le r_k$, we consider the disjoint intervals $I_i = [(i-1)/k, i/k)$ and a final "reserve" interval $I^* = [r_k/k, 1]$ that is added to complete the partition of $[0,1]$. Next, we let $\nu(1)$ be the first integer such that

$$\mathcal{S}_1 := \{X_1, X_2, \ldots, X_{\nu(1)}\} \cap I_1$$

has cardinality n, and for each $i > 1$ we define $\nu(i)$ to be least integer greater $\nu(i-1)$ for which the set $\mathcal{S}_i := \{X_{\nu(i-1)+1}, X_{\nu(i-1)+2}, \ldots, X_{\nu(i)}\} \cap I_i$ has cardinality n. By Wald's lemma and (41) we have

$$\mathbb{E}[\nu(r_k)] = nkr_k \quad \text{where } r_k = \lfloor k - k^{2/3} \rfloor. \tag{42}$$

Now, for each $1 \le i \le n$, we run the *optimal fixed horizon* sequential selection algorithm on $\mathcal{S}_i$, and we let $L(n,i)$ be the length of the subsequence that we obtain. The random variables $L(n,i)$, $1 \le i \le r_k$, are independent, identically distributed, and with mean equal to $\ell(n)$. We then set

$$\mathcal{L}(n, r_k) = L(n,1) + L(n,2) + \cdots + L(n, r_k),$$

and we note that if $\mathcal{L}(n, r_k) \ge N_k$, for N_k as defined in (41), then we have extracted an increasing subsequence of length at least N_k; in this case, we halt the procedure.

On the other hand if $\mathcal{L}(n, r_k) < N_k$, we need to send in the reserves. Specifically, we recall that $I^* = [r_k/k, 1]$ and we consider the post-$\nu(r_k)$ reserve subsequence

$$\mathcal{S}^* := \{X_i : i > \nu(r_k) \text{ and } X_i \in I^*\}.$$

We now rescale the elements of $\mathcal{S}^*$ to the unit interval, and we run the optimal *time-focused* algorithm on $\mathcal{S}^*$ until we get an increasing sequence of length N_k. If we let $R(n,k)$ denote the number of observations from $\mathcal{S}^*$ that are examined in this case, then we have $\mathbb{E}[R(n,k)] = \beta(N_k)$ by the definition of β. Finally, since I^* has length at least $k^{-1/3}$, the expected number of elements of $\{X_i : i > \nu(r_k)\}$ that need to be inspected before we have selected our increasing subsequence of length N_k is bounded above by $k^{1/3}\beta(N_k)$.

The second phase of our procedure may seem wasteful, but one rarely needs to use the reserve subsequence. In any event, our procedure does guarantee that we find an increasing subsequence of length N_k in a finite amount of time T_k. By (42) and the upper bound $k^{1/3}\beta(N_k)$ on the incremental cost when one needs to use the reserve subsequence, we have

$$\beta(N_k) \le \mathbb{E}[T_k] \le knr_k + \{knr_k + k^{1/3}\beta(N_k)\}\mathbb{P}(\mathcal{L}(n, r_k) < N_k), \tag{43}$$

where, as noted earlier, the first inequality comes from the definition of β.

The summands of $\mathcal{L}(n, r_k)$ are uniformly bounded by n and $\mathbb{E}[\mathcal{L}(n, r_k)] = r_k \ell(n)$, so by the definition (41) of N_k and r_k we see from Hoeffding's inequality that

$$\begin{aligned} \mathbb{P}(\mathcal{L}(n, r_k) < N_k) &\le \mathbb{P}\Big(\mathcal{L}(n, r_k) - \mathbb{E}[\mathcal{L}(n, r_k)] < -(k^{2/3} - 1)\ell(n)\Big) \\ &\le \exp\{-A_n k^{1/3}\}, \end{aligned} \tag{44}$$

for constants A_n, K_n, and all $k \ge K_n$. The exponential bound (44) tells us that for each n there is a constant C_n such that the last summand in (43) is bounded by C_n for all k. By the bounds (2) of Theorem 1 we have $\beta(N_k) = (1/2)N_k^2 + O(N_k \log N_k)$, and by (41) we have

$$N_k = (k - 2k^{2/3})\ell(n) + O(1), \quad r_k = k - k^{1/3} + O(1),$$

so, in the end, our estimate (43) tell us

$$\frac{1}{2}\ell^2(n)\{k^2 - 2k^{5/3} + 4k^{4/3}\} \le k^2 n + o_n(k^2).$$

When we divide by k^2 and let $k \to \infty$, we find $\ell(n) \le \sqrt{2n}$, just as we hoped.

Lower Bounds for $\ell(n)$ and the Duality Gap.

One can use the time-focused tools to get a lower bound for $\ell(n)$, but in this case the slippage, or duality gap, is substantial. To sketch the argument, we first let T_r denote the time required by the optimal time-focused selection policy to select r values. We then follow the r-target time-focused policy. Naturally, we stop if we have selected r values, but if we have not selected r values by time n, then we quit, no matter how many values we have selected. This suboptimal strategy gives us the bound $r\mathbb{P}(T_r \le n) \le \ell(n)$, and from this bound and Chebyshev's inequality, we then have

$$r\{1 - \mathrm{Var}(T_r)/(n - \mathbb{E}[T_r])^2\} \le \ell(n). \tag{45}$$

If we then use the estimates (2) and (5) for $\mathbb{E}[T_r]$ and $\mathrm{Var}[T_r]$ and optimize over r, then (45) gives us the lower bound $(2n)^{1/2} - O(n^{1/3})$. However in this case the time-focused bounds and the duality argument leave a big gap.

Earlier, by different methods — and for different reasons — Rhee and Talagrand (1991) and Gnedin (1999) obtained the lower bound $(2n)^{1/2} - O(n^{1/4}) \le \ell(n)$. Subsequently, Bruss and Delbaen (2001) studied a continuous time interpretation of the online increasing subsequence problem where the observations are presented to the decision maker at the arrival times of a unit-rate Poisson process on the time interval $[0, t)$, and, in this new formulation, they found the stunning lower bound $\sqrt{2t} - O(\log t)$. Much later, Arlotto, Nguyen and Steele (2014) showed by a de-Poissonization argument that the lower bound of Bruss and Delbaen (2001) can be used to obtain

$$\sqrt{2n} - O(\log n) \le \ell(n) \qquad \text{for all } n \ge 1$$

under the traditional discrete time model for sequential selection. Duality estimates such as (45) are unlikely to recapture this bound.

7. Observations, Connections, and Problems

A big challenge that remains is to determine the asymptotic distribution of τ_n^*, the time at which one completes the selection of n increasing values by following the unique optimal policy π^* that minimizes the expected time $\mathbb{E}[\tau_n^*] = \beta(n)$. By Theorems 1 and 2 we know the behavior of the mean $\mathbb{E}[\tau_n^*]$ and of the variance $\mathrm{Var}[\tau_n^*]$ for large n, but this does not bring one as close to a distributional limit theorem as one might hope. The crux of the problem is that τ_n^* may be substantially influenced by the time to achieve the last few selections, so anything resembling a central limit theorem for τ_n^* seems unlikely.[1] In a way this is ironic, since for the online size-focused selection problem, two central limit theorems are available. The first was obtained by Bruss and Delbaen (2004) in the context of the model introduced in Bruss and Delbaen (2001), and the second was obtained by Arlotto, Nguyen and Steele (2014) in the traditional context of the online size-focused selection problem that was described in Section 6. The proof of the second CLT depends on a

[1]We are pleased to thank Svante Janson for this last observation.

direct argument using martingales, and it does not appeal to a de-Poissonization argument.

Another natural challenge concerns unimodal subsequence selection problems. In the corresponding size-focused problem, the random variable of interest is

$$U_n^o(\pi_n) = \max\{k : X_{\tau_1} < X_{\tau_2} < \cdots < X_{\tau_t} < X_{\tau_{t+1}} < \cdots < X_{\tau_k},\ \text{where}\\ 1 \le \tau_1 < \tau_2 < \cdots < \tau_k \le n\},$$

and where each τ_k is a stopping time. Here, Arlotto and Steele (2011) found that one has

$$\max_{\pi_n} E[U_n^o(\pi_n)] \sim 2\sqrt{n}, \qquad \text{as } n \to \infty.$$

The analogous time-focused selection problem is again easy to pose, but, its analysis is not easy. In particular, the formulation of a useful analog of Lemma 3 seems problematical.

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