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EFFICACY OF SPACEFILLING HEURISTICS IN EUCLIDEAN COMBINATORIAL OPTIMIZATION

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This note sharpens and generalizes an inequality of Platzman and Bartholdi on the ratio of the cost of the path provided by the spacefilling heuristic to the cost of the optimal path through n points in $\mathbb{R}^d$.

traveling salesman problem * spacefilling curves * Lipschitz mappings * dimension increasing mappings

1. Introduction

Bartholdi and Platzman [1] recently reviewed how one can build heuristics for many problems of combinatorial optimization in Euclidean space by using spacefilling curves that have easily computed inverses. The purpose of this note is to examine the efficacy of the spacefilling heuristic. Specifically, we will sharpen and generalize a bound of Platzman and Bartholdi [8] on the ratio of the length of the tour provided by the spacefilling heuristic to the length of the optimal tour.

The spacefilling heuristic for the traveling salesman problem (TSP) in $\mathbb{R}^d$ is based on a map ϕ from [0, 1] onto $[0, 1]^d$ that is Lip_α with $\alpha = 1/d$, i.e.

$$|\phi(s) - \phi(t)| \leqslant c|s-t|^{1/d}$$

for a constant c and all $0 \leqslant s \leqslant t \leqslant 1$. We should note that it is not possible to have a surjection of [0, 1] onto $[0, 1]^d$ that is in Lip_β for $\beta > 1/d$. This follows from the fact that the union of sets $\phi([i/n, (i+1)/n])$, $0 \leqslant i < n$, must cover $[0, 1]^d$, and for large n $[0, 1]^d$ is not the union of n balls of radius $cn^{-\beta}$ unless $\beta \leqslant 1/d$.

In the positive direction, Milne [6] observed that many of the classical spacefillings curves are in $\text{Lip}_{1/d}$. Moreover, several of those curves are also measure preserving, i.e. for each measurable $A \subset [0, 1]^d$, we have

$$\lambda_1(\phi^{-1}(A)) = \lambda_d(A),$$

where λ_1 and λ_2 denote Lebesgue measures in $\mathbb{R}$ or $\mathbb{R}^d$, respectively.

For a measure preserving Lip_α mapping ϕ from [0, 1] onto $[0, 1]^d$ to be algorithmically effective, one also needs for each $x \in [0, 1]^d$ to be able to quickly compute a $t \in [0, 1]$ such that

$$t \in \phi^{-1}(x).$$

A small but sticky point is that by the invariance of domain theorem (Dugundji [3]), ϕ cannot be one-to-one. Thus, we have to be content with computing *some* point in the pre-image. Since this note deals with the solution efficacy rather than the algorithmic efficiency of the spacefilling heuristic, we will not consider the computation of ϕ^{-1}.

For the spacefilling heuristic the most pressing issue is to obtain a bound on the ratio of the length of the tour produced by the spacefilling curve to the length of the optimal tour. For the TSP in $\mathbb{R}^2$, Platzman and Bartholdi [8] provided a bound of order O(log n), and they conjectured the existence of a uniform bound independent of n. This conjecture has recently been answered in the negative by Bertsimas and Grigni [2] by providing an example that shows that in $\mathbb{R}^2$ the ratio can be as large as $c \log n$ for $c > 0$.

The following theorem complements the results of Platzman and Bartholdi [8] by providing an explicit upper bound on the efficacy ratio of the spacefilling heuristic in $\mathbb{R}^d$ for all $d \geqslant 2$. Moreover, the proof given here makes clear that one does not require any detailed properties of the spacefilling curve in order to provide ratio bounds. All one needs is that the curve is measure preserving and as smooth as feasible.

Theorem 1.1. *Let ϕ be a measure preserving transformation of* [0, 1] *onto* $[0, 1]^d$, *that is Lipschitz of order* $\alpha = 1/d$, *i.e.*

$$|\phi(x) - \phi(y)| \leqslant c|x - y|^{1/d} \tag{1.1}$$

for some c and all $x, y \in [0, 1]$. *If H_n is the length of the path through a set of n points $S \subset [0, 1]^d$ that is constructed using the spacefilling heuristic based on ϕ, then for $n \geqslant 2$*

$$H_n \leqslant \{1 + \omega_{d-1}c^d \log(m/\bar{e})\} L_n + \omega_d c^d m, \tag{1.2}$$

where L_n is the length of the optimal path through $\{x_1, x_2, \dots, x_n\}$, *m is the length of the longest edge in the heuristic path, $\bar{e}$ is the average length of the edges in the optimal path, and ω_k denotes the volume of the unit ball in $\mathbb{R}^k$ for $k = d - 1$ or d.*

Corollary 1.2. *We have for all $n \geqslant 2$ that*

$$H_n \leqslant (1 + \omega_{d-1}c^d + \omega_d c^d \log n) L_n. \tag{1.3}$$

2. Proof of main result

We require a preliminary geometric inequality of Esterman [4] that has recently become of interest in mathematical statistics, cf. Naiman [7], and Johnstone and Siegmund [5].

Lemma 2.1. *If $T(x, C) \subset \mathbb{R}^d$ is the set of all points within distance x of the rectifiable curve C, then for all $x \geqslant 0$ we have*

$$\mu(T(x, C)) \leqslant \omega_{d-1}x^{d-1}L + \omega_d x^d, \tag{2.1}$$

where ω_d is the volume of the unit ball in $\mathbb{R}^d$ and L is the length of C.

We begin the proof of Theorem 1.1 by assuming that the points of S are labeled in the order they appear on the heuristic tour, so we write $S = \{x_1, x_2, \dots, x_n\}$, and we assume for each $1 \leqslant i \leqslant n$ there is a $t_i \in [0, 1]$ such that $t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n$ with $x_i = \phi(t_i)$. Next, for $u > 0$ we introduce two subsets of $\{1, 2, \dots, n-1\}$ by

$$U(u) = \{i : |t_{i+1} - t_i| > u, 1 \leqslant i < n\},$$

and

$$V(u) = \{i : |\phi(t_{i+1}) - \phi(t_i)| > u, 1 \leqslant i < n\}.$$

The benefit of introducing $V(u)$ is that its cardinality $g(u)$ gives a formula for H_n:

$$H_n = \int_0^m g(u)\, \mathrm{d}u, \tag{2.2}$$

where

$$m = \max_{i \leqslant i < n} |\phi(t_{i+1}) - \phi(t_i)|.$$

Thus, we need to find a bound for $g(u)$, and our method for achieving this will depend on first obtaining a bound on $f(u) = |U(u)|$, the cardinality of $V(u)$.

For $i \in U(u)$ the intervals $[t_i, t_i + u]$ do not intersect, so if we set

$$A_i = \phi([t_i, t_i + u]),$$

then, since ϕ preserves measure, we have

$$uf(u) = \sum_{i \in U(u)} \lambda_d(A_i) = \lambda_d\Big(\bigcup_{i \in U(u)} A_i\Big). \tag{2.3}$$

Now we use Esterman's inequality (2.1) to bound the right hand side of (2.3). Letting C_n be an optimal tour of $\{x_1, x_2, \dots, x_n\}$ with length L_n, we see by (1.1) and the fact that each x_i is somewhere on the path C_n that

$$A_i \subset T(cu^{1/d}, C_n), \quad 1 \leqslant i \leqslant n. \tag{2.4}$$

Hence, by (2.2) and Esterman's inequality we have

$$uf(u) = \mu\Big(\bigcup_{i \in U(u)} A_i\Big) \leqslant \mu(T(cu^{1/d}, T_n)) \leqslant \omega_{d-1}c^{d-1}u^{(d-1)/d}L_n + \omega_d c^d u.$$

We thus obtain the bound

$$f(u) \leqslant \omega_{d-1}c^{d-1}u^{-1/d}L_n + \omega_d c^d. \tag{2.5}$$

Now, to bound g we note that for $i \in V(u)$ inequality (1.1) implies

$$c|t_i - t_{i+1}|^{1/d} \geqslant |\phi(t_{i+1}) - \phi(t_i)| \geqslant u,$$

hence we have

$$V(u) \subset U(c^{-d}u^d), \tag{2.6}$$

and thus $g(u) \leqslant f(c^{-d}u^d)$. By (2.5) and (2.6) we find our basic bound

$$g(u) \leqslant \omega_{d-1}c^d u^{-1}L_n + \omega_d c^d. \tag{2.7}$$

For any $0 < \alpha < m$, we can apply the trivial bound $g(u) \leqslant n-1$ for $u \in [0, \alpha]$ and apply (2.7) for $u \in [\alpha, m]$; so, when we integrate in (2.7), we find

$$H_n \leqslant \alpha(n-1) + \omega_{d-1}c^d L_n \log(m/\alpha) - \omega_d c^d(m-\alpha). \tag{2.8}$$

Finally, since $L_n \leqslant H_n \leqslant (n-1)m$ we have for $\alpha = L_n/(n-1) = \bar{e}$ that $\alpha \in [0, m]$, so we can let $\alpha = \bar{e}$ in (2.7) to find (1.2). To see that (1.3) follows from (1.2) we use the very crude bound $m \leqslant L_n$ and $\bar{e} = L_n/(n-1)$.

3. Concluding remarks

The proof of Theorem 1.1 given here uses several ideas from Bartholdi and Platzman [1], where it was proved that $H_n/L_n = O(\log n)$. The present approach sharpens and generalizes that bound by making systematic use of Esterman's inequality and the bound (2.3).

A potentially useful feature of inequality (1.2) is the presence of the ratio

$$m/\bar{e} = (n-1)m/L_n.$$

The only time the spacefilling heuristic can perform badly is when the longest edge in the heuristic path is much larger than the average edge in the optimal. In the example of Bertsimas and Grigni [2], the critical ratio $m/\bar{e}$ is of order n.

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Note added in proof

The spacefilling curve heuristic for the TSP goes back at least to unpublished work of S. Kakutani in 1966. The early history of the idea is discussed by R. Adler in *the Collected Works of S. Kakutani, Vol. II,* R.R. Kalman (ed.), Birkhauser, Boston, 1986, p. 444.

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