and

$$D_n(A) = \max\{k\colon Y_{i_1} > Y_{i_1} > \cdots > Y_{i_k} \text{ with } X_{i_1} < X_{i_2} < \cdots < X_{i_k},\ X_{i_j} \in A \text{ and } i_j \in [1, 2, \ldots, n]\}.$$

Next set

$$U_n = \max_{0 \leq t \leq 1} \{\max(I_n([0, t]) + D_n([t, 1]), D_n([0, t]) + I_n([t, 1]))\}.$$

The desired proof will be obtained by applying known results to the random variable U_n. To begin it is easy to check that

$$EU_n = l(n).$$

Next we note that by the work of Hammersley [2] and Kesten [3] that almost surely and in L^1 we have the limits

$$\lim_{n\to\infty} I_n(A)/\sqrt{n} = C'\sqrt{\lambda(A)} \quad \text{and} \quad \lim_{n\to\infty} D_n(A)/\sqrt{n} = C'\sqrt{\lambda(A)}$$

where $\lambda(A)$ is the Lebesgue measure of $A \subset [0, 1]$, and C' is a universal constant. The work of Logan and Shepp [9] and Vershik and Kerov [5] established that $C' = 2$.

For any N and $1 \leq k \leq N$ we define

$$U_n^N(k) = \max[I_n(0, k/n) + D_n((k-1)/N, 1), D_n(0, k/N) + I_n((k-1))/N, 1)]$$

and

$$U_n^N = \max_{1 \leq k \leq N} U_n^N(k).$$

Clearly, for all N, $U_n \leq U_n^N$ and by the above mentioned limit results we have

$$\lim_{n\to\infty} U_n^N/\sqrt{n} = 2 \max_{1 \leq k \leq N} (\sqrt{k/N} + \sqrt{(N-k+1)/N}),$$

where the limit is almost sure and in L^1. The arbitrariness of N then shows

$$\limsup_{n\to\infty} U_n/\sqrt{n} \leq 2 \max_{0 \leq t \leq 1} (\sqrt{t} + \sqrt{1-t}) = 2\sqrt{2} \quad \text{a.s.},$$

so by Fatou's lemma we get

$$\limsup_{n\to\infty} l(n)/\sqrt{n} \leq 2\sqrt{2}.$$

For the opposite direction note the trivial bound

$$U_n \geq I_n([0, \tfrac{1}{2}]) + D_n[\tfrac{1}{2}, 1])$$

so

$$\liminf_{n\to\infty} l(n)/\sqrt{n} \geq \liminf_{n\to\infty} E(I_n[0, \tfrac{1}{2}] + D_n[\tfrac{1}{2}, 1]) = 2\sqrt{2}$$

which completes the proof.

3. The generalization

Instead of allowing the subsequence to make "one turn" as in the unimodal case, one can consider subsquences which make k turns. Explicitly, let $l_k(n)$ be the expected length of the longest subsequence S of a random permutation with the following property:

S can be decomposed into $k+1$ segments which are monotone and which alternate between increasing and decreasing.

The method of the preceeding section can be used easily to show

$$\lim_{n\to\infty} l_k(n)/\sqrt{n} = 2\sqrt{k+1};$$

all one has to do is define the proper analogue $U_n(k)$ of U_n and argue as before. One should also note that the preceding bounds also prove the almost sure and L^1 convergence of $U_n(k)/\sqrt{n}$ to $2\sqrt{k+1}$.

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