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GROWTH RATES FOR MONOTONE SUBSEQUENCES[1]

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ABSTRACT. The growth rate of the largest monotone subsequence of a uniformly distributed sequence is obtained. For $a_n = n\alpha \bmod 1$ with α algebraic irrational the exponent of growth is found to be precisely the same as for a random sequence.

1. Introduction. A well-known result of Erdös and Szekeres [1] states that any sequence of n real numbers contains a monotone subsequence with at least $n^{1/2}$ elements. More recently, Hammersley [2] proved that if $l_n = l_n(a_1, a_2, \ldots, a_n)$ is the order of the largest increasing subsequence of $a_1, a_2, \ldots, a_n$, and the a_i are chosen independently with the uniform distribution on [0, 1], then

$$\lim_{n\to\infty} n^{-1/2} l_n = C, \tag{1}$$

where C denotes a constant and the convergence is in probability. This result was strengthened by Kesten [4] to provide almost sure convergence, and Logan and Shepp [6] proved that $C \geqslant 2$. Our objective here is to provide results like (1) for sequences which are uniformly distributed in [0, 1], but which are not random. Of particular interest to us is the sequence $a_n = n\alpha \bmod 1$ where α is an algebraic irrational.

2. Uniformly distributed sequences. We will denote by $1_{[a,b)}(x)$ the indicator function of the interval $[a, b)$ and will say a sequence (a_n) is uniformly distributed in [0, 1] provided for all $0 \leqslant a < b \leqslant 1$,

$$\lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} 1_{[a,b)}(a_i) = b - a.$$

The best one can say about the growth rate of l_n for a general uniformly distributed sequence is the following:

THEOREM 1. *If (a_n) is uniformly distributed, then*

$$\lim_{n\to\infty} n^{-1} l_n = 0. \tag{2}$$

PROOF. Let A and n be positive integers and for $0 \leqslant i \leqslant A - 1$ and

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$0 \leqslant j \leqslant A - 1$ let

$$S_{ij} = \{k: 1 \leqslant k \leqslant n, iA^{-1} \leqslant a_k < (i+1)A^{-1},$$
$$jnA^{-1} + 1 \leqslant k \leqslant (j+1)nA^{-1}\}.$$

By $|S_{ij}|$ we denote the cardinality of S_{ij} and we set $g(n) = \max_{i,j}|S_{ij}|$. If n tends to infinity along the subsequence $n = \gamma A$, $\gamma = 1, 2, \ldots,$ then $g(n)/n$ is easily seen to converge to A^{-2} by the uniform distribution of (a_n).

Next let $S = \{i_1 < i_2 < \cdots < i_s\}$ be any subsequence of $1, 2, \ldots, n$ such that $a_{i_1} \leqslant a_{i_2} \leqslant \cdots \leqslant a_{i_s}$. We note that S intersects at most $2A - 1$ of the S_{ij}. (One can identify $a_1, a_2, \ldots, a_n$ with its graph in $\{1, 2, \ldots, n\} \times [0, 1]$ and view the S_{ij} as "boxes.") This observation yields the inequality $|S| \leqslant 2Ag(n)$, and since $l_n \leqslant |S|$ we have $\overline{\lim}_{n\to\infty} l_n/n \leqslant 2/A$ provided the limit is taken along the subsequence $n = kA$.

For $kA < n < (k+1)A$ we note that

$$l(a_1, a_2, \ldots, a_n) \leqslant l(a_1, a_2, \ldots, a_{Ak}) + l(a_{Ak+1}, \ldots, a_n)$$
$$\leqslant l(a_1, a_2, \ldots, a_{Ak}) + A.$$

This proves

$$\overline{\lim_{n\to\infty}} \frac{l_n}{n} \leqslant \overline{\lim_{k\to\infty}} \frac{(l_{kA} + A)}{kA} \leqslant \frac{2}{A},$$

which completes the proof of (1), since A was an arbitrary positive integer.

3. Results concerning $(n\alpha)$. To show that $l_n = o(n)$ is best possible we do not have to go out of the class of sequences $a_n = n\alpha \bmod 1$.

THEOREM 2. *Let C_n be a sequence of real numbers such that $C_n \to 0$ as $n \to \infty$; then there is a transcendental α such that for $a_n = n\alpha \bmod 1$ we have*

$$n^{-1}l_n \geqslant C_n \quad \textit{for infinitely many } n. \tag{3}$$

PROOF. The proof depends on an elementary lower estimate for l_n in terms of the denominators q_k of the convergents p_k/q_k of α. First we assume $n = q_{k+1}$ and that $\{q_k\alpha\} > 0$, where $\{x\} = x - [x + \frac{1}{2}]$. For $j = Sq_k$ the sequence $j\alpha$ with $S = 1, 2, \ldots, [q_{k+1}/q_k]$ can be viewed as making small positive steps, so we have the lower bound

$$l_n \geqslant \min(1/\{q_k\alpha\}, q_{k+1}/q_k). \tag{4}$$

By the standard theory of continued fractions (e.g., [3, p. 9]) we have $|\{\alpha q_{k+1}\}| < 1/q_{k+1}$, so (4) implies $l_n \geqslant q_{k+1}/q_k$. Since $C_n \to 0$ we can choose q_k which go to infinity as rapidly as we like such that $1/q_k \geqslant C_t$ for $t = q_{k+1}$. In particular, we may require q_k to grow rapidly enough to insure that α is transcendental. Finally, we note that if the condition $\{q_k\alpha\} > 0$ is not met by infinitely many k, we need only replace α by $1 - \alpha$. This will then complete the proof.

There is a more precise result which can be proved if α is algebraic. To state it succinctly, we let l_n' denote the order of the largest monotone (increasing or decreasing) subsequence of $a_1, a_2, \ldots, a_n$.

THEOREM 3. *If $a_n = n\alpha \bmod 1$ where α is an algebraic irrational, then*

$$\lim (\log l'_n)/(\log n) = 1/2. \tag{5}$$

PROOF. We must obtain quantitative versions of the estimates used in Theorem 1. To begin, for $0 \leqslant i \leqslant n-1$ and $0 \leqslant j \leqslant n-1$ we let

$$S_{ij} = \{a_k : i/n \leqslant a_k < (i+1)/n, jn+1 \leqslant k \leqslant (j+1)n\}$$

and observe that

$$\max_{i,j} |S_{ij}| \leqslant \max_{0 \leqslant j \leqslant n-1} \{1 + 2nD_n^j\}, \tag{6}$$

where

$$D_n^j = \sup_{0 \leqslant x \leqslant 1} \left| n^{-1} \sum_{k=jn+1}^{(j+1)n} 1_{[0,x)]}(a_k) - x \right|.$$

Also, if $S = \{a_{i_1}, a_{i_2}, \ldots, a_{i_s}\}$ is any monotone subsequence of $\{a_1, a_2, \ldots, a_{n^2}\}$, we know S intersects at most $2n-1$ of the S_{ij}. Thus, we have

$$n \leqslant l'_{n^2} \leqslant 2n \max_{i,j} |S_{ij}|, \tag{7}$$

where the first inequality follows from the Erdös-Szekeres theorem mentioned in the introduction.

Since the sets $\{(jn+1)\alpha, (jn+2)\alpha, \ldots, (j+1)n\alpha\}, j = 0, 1, \ldots, n-1$, are translates of $\{\alpha, 2\alpha, \ldots, n\alpha\}$, we have

$$\max_{0 \leqslant j \leqslant n-1} D_n^j = O(D_n^1). \tag{8}$$

By the Thue-Siegel-Roth theorem [5, pp. 122–124] we know that $D_n = D_n^1 = O(n^{-1+\varepsilon})$ for all $\varepsilon > 0$. This fact, with (7) and (8), yields

$$\lim_{n \to \infty} (\log l'_{n^2})/(\log n) = 1. \tag{9}$$

For the final step choose n so that $n^2 \leqslant j < (n+1)^2$ and note $l'_{n^2} \leqslant l'_j \leqslant l_{n^2} + 2n$. By the bounds on j and the limit in (9), one completes the proof with a brief computation.

There are two corollaries of the proof of Theorem 3.

COROLLARY. 1. *If α is an irrational for which $D_n = O(n^{-1+\varepsilon})$ for all $\varepsilon > 0$, then* (5) *holds. In particular, this is the case if α is of finite type* 1.

COROLLARY 2. *For all α except a set of measure* 0, *one has* (5).

The proof of Corollary 2 depends only on the fact that $D_n = O(n^{-1+\varepsilon})$ for all $\varepsilon > 0$ and almost every α. (For more precise results on D_n, see Niederreiter [7]).

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