Japanese Journal of Applied Physics
Vol. 25, No. 7, July, 1986, pp. 1136–1137

Hasegawa's Three Point Method for Determining Transient Time Constants

J. Michael Steele

School of Engineering and Applied Science, Princeton University, Princeton NJ 08544, U.S.A.

(Received February 24, 1986; accepted for publication April 26, 1986)

In many physical contexts it is important to estimate the time constant for a transient which is represented by a curve with exponential decay. The three point method for estimating such constants is studied here with the aim of understanding the influence of measurement error. The three point method is found to depend delicately on the choice of the middle (or *design*) point, to have a small positive bias, and to have a large standard error relative to the measurement errors.

The purpose of this note is to remark on the statistical properties of a simple method (the Three Point Method) which was given by Hasegawa[1)] for estimating the time constant associated with phenomena whose transients can be expressed as an exponential function.

Methods which speak directly to the physical phenomena of transients and which come before Hasegawa's Three Point Method include: plotting the logratihm of the capacitance change against time,[2)] the deep transient spectroscopy (DLTS) method introduced by Lang,[3)] the isothermal capacitance transient spectroscopy (ICTS) method which essentially plots the differential of the capacitance and which calculated the time constant from the maximum of the differential,[4)] and the Fourier Series method more recently introduced by Ikeda and Takaoka.[5)]

The Three Point Method has a compelling analytic simplicity and it addresses a recurring physical problem. It is therefore reasonable to examine the method from the view point of the statistical properties of mean square error and of bias. In the next section the so-called delta method is applied to give closed analytical expression for the approximate variance of the three point estimator. The third section reports the results of a Monte Carlo study. The main points to be made by that study are (1) the delta method estimate of variance is justified as a good approximation of the true variance (2) the three point method has relative standard error which is (even in the best cases) is almost 20 times as large as the standard error of the individual observations, (3) the three point method has a positive bias for all reasonable choices of the design parameter and (4) the bias of the three point is not intolerably large (ranging from 0.5 to 2.0 percent for reasonable choices of the design parameter).

Capacitance change $C(t)$ is commonly assumed[1–5)] to be expressed by

$$C(t)=C_\infty(1-N\exp(-\alpha t)) \tag{1}$$

where C_∞ is the capacitance at $t=\infty$, N is a constant (which in certain instances can be explicitly related to the shallow donor and trap densities), and α is the emission rate.

To estimate α from the value of $C(t)$ with the Three Point Method, one first chooses a *design parameter s* and then sets $t_1=s/2$, $t_2=s$, and $t_3=3s/2$. Hasegawa[1)] observed that a good choice of s is $\tau=\alpha^{-1}$, the time constant. Since τ is not assumed to be known one can begin with a crude estimate of τ or a subjective *a priori* estimate. The three point estimate of α is given by letting $\Delta C_1=C(t_2)-C(t_1)$, $\Delta C_2=\Delta C(t_3)-\Delta C(t_2)$, $\Delta t=t_2-t_1=t_3-t_2$ and observing by simple arithmetic that one then has

$$\alpha=\frac{1}{\Delta t}\ln\left[\frac{\Delta C_1}{\Delta C_2}\right] \tag{2}$$

To model the measurement error associated with a method like this, one typically assumes that the actual observations $\hat{C}(t_i)$ are expressible as $C(t_i)+\varepsilon_i$ where the ε_i are independent normally distributed random variables with mean 0 and variance σ^2. Setting $\Delta\hat{C}_i=\hat{C}(t_{i+1})-\hat{C}(t_i)$ one can estimate the effect of modest noise on α by the delta of first linearly approximating $\hat{\alpha}$ as a function of the noise and the true α,

$$\begin{aligned}\hat{\alpha}&=\frac{1}{\Delta t}\ln\left[\frac{\Delta\hat{C}_1}{\Delta\hat{C}_2}\right]\\&\approx\frac{1}{\Delta t}\left\{\ln\Delta C_1+\frac{\varepsilon_2-\varepsilon_1}{\Delta C_1}-\ln\Delta C_2-\frac{\varepsilon_3-\varepsilon_2}{\Delta C_2}\right\}\\&\approx\alpha+\frac{1}{\Delta t}\left\{\frac{-\varepsilon_1}{\Delta C_1}+\varepsilon_2\left[\frac{1}{\Delta C_1}+\frac{1}{\Delta C_2}\right]-\frac{\varepsilon_3}{\Delta C_2}\right\}\end{aligned} \tag{3}$$

and then obtaining the so-called delta method estimate of variance by taking the variance of the linear approximation,

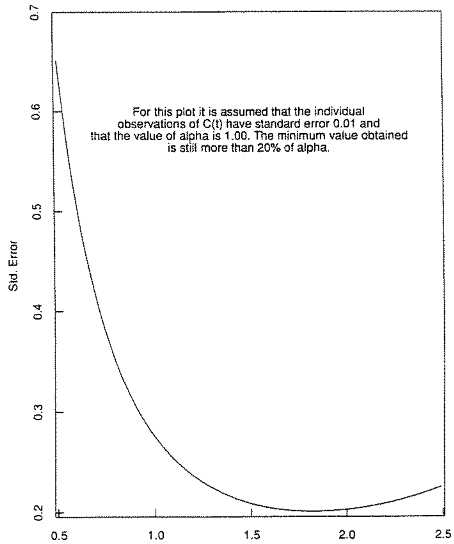

Fig. 1. Delta Method Estimate of Standard Errors as a function of the Design Parameter S.

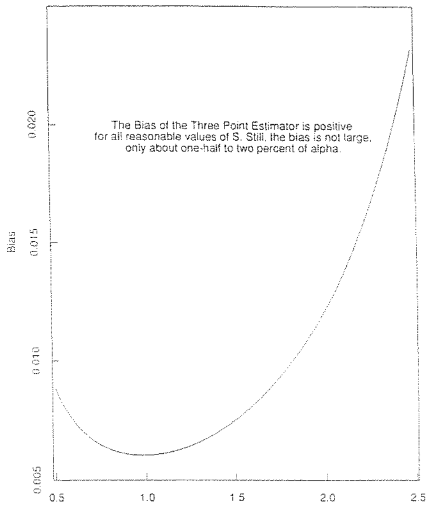

Fig. 2. Bias of the 3 Point Estimator as a function of the Design Parameter S.

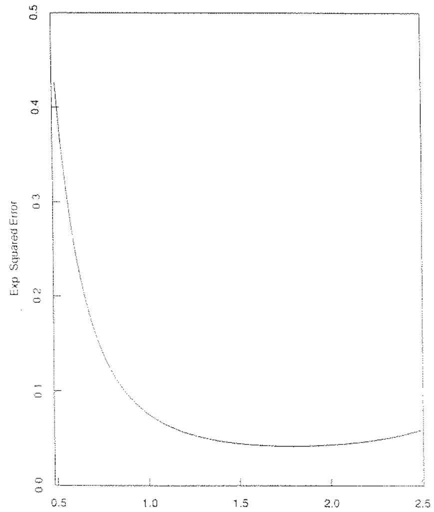

Fig. 3. The minimum value of the expected squared error is attained outside the range where bias is smallest.

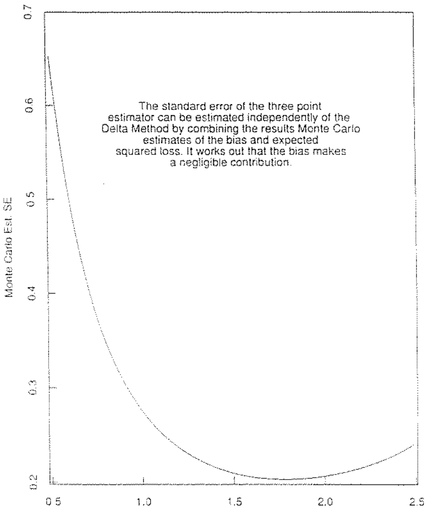

Fig. 4. Monte Carlo Estimate of Standard Errors as a function of Design Parameter.

$$Var(\hat{\alpha}) \approx \frac{\sigma^2}{(\Delta t)^2}\left\{(\Delta C_1)^{-2}+\frac{(C_3-C_1)^2}{(\Delta C_1 \Delta C_2)^2}+(\Delta C_2)^{-2}\right\} \qquad (4)$$

For the sake of comparisons one can take $C_\infty = N = \alpha = 1$ in (3) and plot the approximate variance given by (4) as a function of the design parameter s. For the sake of specificity we take σ to be 0.01 in the plots which follow. The assumption that the measurement error is typically as much as one percent of the parameter being estimated is not critical and the essence of the plots is unchanged by varying that assumption. As the results of the Monte Carlo estimates will indicate, delta method estimate of variance is quite good through almost all its range.

Under the same conditions as above, the behavior of $\hat{\alpha}$ by Monte Carlo methods. We let the design parameter s vary from 0.5 to 2.5 and calculated one thousand values of $\hat{\alpha}$ for each value of s. The mean these of these one thousand calculated values of $\hat{\alpha}$ gives a reasonable estimate of the expected value of $\hat{\alpha}$. When $\alpha = 1.0$ is subtracted from those means one then has an estimate of the bias of $\hat{\alpha}$ as function of the design parameter s. The plot of these biases given in Fig. 2 show that $\hat{\alpha}$ consistently exhibits a reasonably small bias. One consequence of this observation is that with out correcting for that bias, no number of replications of $\hat{\alpha}$ can lead to a consistent estimate of α. In addition to the Monte Carlo estimate of bias the same design was used to provide an independent estimate of the standard error. The mean sum of squared errors was calculated for one thousand values of $\hat{\alpha}$ at each design value s. These values are given in Fig. 3. Since the variance of $\hat{\alpha}$ equals the sum of the expected squared error loss and the the square of the bias, the simulations of Figs. 2 and 3 can be combined to give an independent estimate of the standard error of $\hat{\alpha}$. This estimate is given in Fig. 4 and shows excellent agreement with the delta method results of Fig. 1.

The Hasegawa Three Point Method for estimating time constants of transient phenomena has been examined in the context of noisy observations. Analytical estimates of the variance and Monte-Carlo analyses of the mean and variance provided information of the dependence of the three point critical choice of the middle point (design parameter). While the analytical elegance of the method can be commended, the practical applications of the Three Point Method seems best constrained to those situations where the time constant is known *a priori* with some precision, the coefficient of variation of the observations is quite small, and observations at more than three time points is costly or impractical.

This work was supported in part by NSF grant DBS-8414069.

References

1) F. Hasegawa: Jpn. J. Appl. Phys. **24** (1985) 1356.
2) R. R. Senechal and J. Basinski: J. Appl. Phys. **39** (1968) 4581.
3) D. V. Lang: J. Appl. Phys. **45** (1974) 3023.
4) J. Okushi and Y. Tokumaru: *Proc. 12th Conf. Solid State Devices, Tokyo, 1980*, Jpn. J. Appl. Phys. **20** (1981) Suppl. 20-1, p. 261.
5) K. Ikeda and H. Takaoka: Jpn. J. Appl. Phys. **21** (1982) 462.