establish that $\hat{\lambda} = u(\hat{\theta}) \rightarrow S_u^2(\hat{\lambda}) \leq S_u^2(\lambda)$ for all λ in Λ. In the "classical" linear model $E_\theta(\mathbf{X}) = \mathbf{b}(\boldsymbol{\theta}) = \mathbf{B}\boldsymbol{\theta}$ and $\text{Cov}(\mathbf{X}) = \mathbf{A}^{-1}$. However, the assertion above holds even without this particular specification of $\mathbf{b}(\theta)$ and $\mathbf{A}$.

Indeed, as one can see, this invariance property could be claimed for any method of estimation which defines the estimate, if it exists, as the one minimizing the value of a suitable non-negative "distance" function.

REFERENCE

[1] Zehna, Peter W. (1966). Invariance of maximum likelihood estimators. *Ann. Math. Statist.* 37, 744.

When Successes and Failures are Independent a Compound Process is Poisson

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Let N be a random variable which takes on non-negative integer values, and let X be a random variable which takes on values $E_1, E_2, \ldots, E_r$. Now let Y_k denote the number of occurrences of event E_k in N independent trials of the random variable X. If N is Poisson, it has been observed ([1], p. 217) that $Y_1, Y_2, \ldots, Y_r$ are independent. In the case that X is Bernoulli, and E_1 denotes success and E_2 denotes failure this yields the interesting situation that the random variables Y_1 and $Y_2 = N - Y_1$ are independent. It should be noted that the random variables determined by Y_i conditioned on the event $N = n$ need not be independent, but with this precaution one can express the independence of Y_1 and Y_2 by saying that in a Poisson number of trials the number of successes is independent of the number of failures. The theorem of this note forms a converse to the preceding observations.

Theorem. If there exist two of the Y_i which are independent then N must be Poisson.

Proof. Suppose Y_1 and Y_2 are independent, and let $f(s)$ be the generating function for the random variable N. Then letting $p_i = P[X = E_i]$ we have that the generating function for Y_i is $f(1 - p_i + p_i s)$. Calculating the bivariate generating function for Y_1 and Y_2 gives

$$\sum_{k,l} P[Y_1 = k, Y_2 = l] s^k t^l = f(1 - p_1 - p_2 + p_1 s + p_2 t). \qquad (1)$$

Now, by the independence of Y_1 and Y_2 we have

$$f(1 - p_1 - p_2 + p_1 s + p_2 t) = f(1 - p_1 + p_1 s) f(1 - p_2 + p_2 t). \qquad (2)$$

Letting $a = 1 - p_1 + p_1 s$ and $b = 1 - p_2 + p_2 t$ yields the equation

$$f(a + b - 1) = f(a) f(b). \qquad (3)$$

If $f(0) = 0$ then setting $a = b = \frac{1}{2}$ in equation (3) gives $0 = f(\frac{1}{2})$, which is impossible since $f(s)$ is the generating function of a non-negative random variable. Hence we have $f(0) \neq 0$. Now let $g(s) = f(s)/f(0)$. We have

$$g(a)g(b) = f(a)f(b)/[f(0)]^2 = f(a + b - 1)/[f(0)]^2. \qquad (4)$$

Now note that letting $b = 0$ in (3) gives $f(a - 1) = f(a)f(0)$, and then replacing a by $a + b$ gives $f(a + b)f(0) = f(a + b - 1)$. This yields

$$g(a)g(b) = f(a + b)f(0)/[f(0)]^2 = g(a + b). \qquad (5)$$

Since $g(s)$ is monotone (5) has the unique solution $g(s) = e^{\mu s}$ where $\mu \geq 0$ is a constant.

Now we have $g(1) = 1/f(0)$ so $f(0) = e^{-\mu}$, and $f(s) = e^{-\mu + \mu s}$ which is the generating function of the Poisson distribution.

Corollary. If two of the Y_i are independent then all of the Y_i are independent.

REFERENCE

[1] W. Feller, *An Introduction to Probability Theory and Its Applications*, Vol. I, New York (Wiley), 1968 (3rd ed.).

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