The Annals of Probability
1980, Vol. 8, No. 1, 83–95

VECTOR VALUED SUBADDITIVE PROCESSES AND APPLICATIONS IN PROBABILITY[1]

BY N. GHOUSSOUB AND J. MICHAEL STEELE

University of British Columbia and Stanford University

An ergodic theorem is proved which extends the subadditive ergodic theorem of Kingman and the Banach valued ergodic theorem of Mourier. The theorem is applied to several problems, in particular to a problem on empirical distribution functions.

1. Introduction. The subadditive ergodic theorem of Kingman [2, 5, 6, 16, 17, 18, 21] is a nonlinear generalization of the Birkhoff ergodic theorem which, in the ten years of its existence, has become established as one of the most useful results in probability theory. Fifteen years before Kingman's result, Mourier [19] had extended the Birkhoff theorem in an entirely different direction by proving that it is valid in any Banach space. The main objective of this paper is to prove a result which extends both the Kingman and Mourier theorems.

While the simple existence of such a generalization may be of interest, our primary motivation has been the feeling that such a result should be genuinely useful. To expedite possible applications we have outlined in the third section several general procedures which give rise to vector valued subadditive processes. In particular we treat a problem there on empirical distribution functions which seems of independent interest.

A second motivation has been provided by the intimate relationship between the subadditive ergodic theorem and the geometry of the underlying Banach lattice. It is possible, for example, to characterize those spaces isomorphic to L^1 by means of subadditive processes.

Before yielding to the details of Section 2, it seems useful to give a less technical description of our main result. We suppose E is a Banach lattice (i.e., $|x| \leqslant |y| \Rightarrow \|x\| \leqslant \|y\|$), and we will call a sequence (S_n) of Bochner integrable E-valued functions a *subadditive process* provided for all natural n, k we have

$$S_{n+k} \leqslant S_n + S_k \circ \theta^n \tag{1.1}$$

where θ is a measure preserving transformation on the probability space $(\Omega, \mathfrak{F}, P)$.

Our key result is that under a mild restriction on E, one has for any positive subadditive process that S_n/n converges in norm with probability one. In particular we are content to note here that E can be L^p, $1 \leqslant p < \infty$, c_0, or essentially any familiar space except l^∞ or $C(K)$.

Received July 14, 1978.
[1] Supported by National Science Foundation Grant MCS77-16974.
AMS 1970 *subject classification.* Primary 60F15.
Key words and phrases. Ergodic theorem, subadditive ergodic theorem, Banach lattice, empirical distributions.

2. Main results. We must first recall some facts from ergodic theory and functional analysis. When equality is assumed to hold in (1.1), (S_n) is called an additive process, and the key result about such processes is the following.

VECTOR VALUED ERGODIC THEOREM. (*Mourier* [19]). *If (S_n) is an additive process valued in any Banach space E then S_n/n converges* a.s. *in norm to $E^I(S_1)$ where I is the σ-field of θ invariant sets. (Here and subsequently E^I will denote the conditional expectation with respect to the σ-field I).*

REAL SUBADDITIVE ERGODIC THEOREM. (*Kingman* [16]). *If (S_n) is a real subadditive process then* $\inf n^{-1}E(S_n) > -\infty$ *implies*

$$\lim S_n/n = \xi \text{ a.s.} \tag{2.1}$$

where

$$\xi = \lim E^I(S_n/n). \tag{2.2}$$

We need to recall some notions about Banach lattices which may not be familiar to probabilists. A Banach lattice E is called countably order complete (COC) provided for any nonempty countable B which is majorized by an element of E that sup B exists. Further, E has order continuous norm (OCN) provided E is COC and every decreasing positive sequence in E is norm convergent. Finally, E is weakly sequentially complete (WSC) if every increasing norm bounded sequence in E is norm convergent. (For background on these notions, one can consult Schaefer [21], pages 54 and 92.)

A sequence (x_n) in E will be called subadditive (superadditive) if $x_{n+k} \leqslant x_n + x_k$ ($x_{n+k} \geqslant x_n + x_k$) for all natural n, k; we start by developing several elementary facts about subadditive sequences in Banach lattices. Finally, we let E' denote the dual of E and let E'_+ denote the positive cone of E'.

LEMMA 1. *In a Banach lattice E with an order continuous norm one has the following properties*:

$$\text{If } 0 \leqslant x_n \leqslant x \text{ and } x_n \text{ converges weakly to } x \text{ then } x_n \text{ converges strongly to } x. \tag{2.3}$$

$$\text{If } 0 \leqslant x_n \leqslant y_n, y_n \text{ converges strongly to } x \text{ and } x_n \text{ converges weakly to the same } x \text{ then } x_n \text{ converges strongly to } x. \tag{2.4}$$

PROOF. This lemma has been established in Heinich [14], but we will provide an alternate proof. By the result of Diestel and Seifert [7] which says that a weakly compact order interval is the range of a countably additive vector measure, and the result of Anantharaman [8], page 270 which says the extreme points of such ranges are denting points we see that x is a denting point of $[0, x]$. Thus we have established the first part of the lemma. To prove the second half we note

$0 \leqslant x_n \leqslant x_n \vee x \leqslant y_n \vee x$ hence for $f \in E'_+$ we have $f(x_n) \leqslant f(x_n \vee x) \leqslant f(y_n \vee x)$ and thus $x_n \vee x$ converges weakly to x. Since $x_n \vee x + x_n \wedge x = x_n + x$ we also see $x_n \wedge x$ converges weakly to x, which implies by (2.3) that $x_n \wedge x$ converges strongly to x. Since $0 \leqslant x_n \wedge x \leqslant x_n \leqslant y_n$ the strong convergence of the bounds on x_n implies (2.4).

PROPOSITION 1. *For a countably order complete Banach lattice E the following conditions are equivalent*:

(2.5) *E has an order continuous norm.*

(2.6) *For every positive subadditive sequence (x_n) in E, the sequence x_n/n is norm convergent to* $\inf(x_n/n)$.

PROOF. If (x_n) is subadditive, $(x_{2^k}/2^k)$ is decreasing and assuming E has an order continuous norm this sequence is norm convergent to $z = \inf_k x_{2^k}/2^k = \inf x_n/n$. Since applying any element of E'_+ to x_n yields a real subadditive sequence we see that (x_n/n) is weak Cauchy. Since this sequence is valued in the weakly compact order interval $[z, x_1]$ it is weakly convergent to z. By Lemma 1 this is enough to imply norm convergence to z.

To prove that (2.6) implies (2.5) it suffices to note that if (y_n) is positive decreasing then $(x_n) = (ny_n)$ is subadditive.

PROPOSITION 2. *For a Banach lattice E, the following conditions are equivalent*:

(2.7) *E is weakly sequentially complete.*

(2.8) *For every superadditive sequence (x_n) in E such that* $\sup\|x_n\|/n < \infty$ *the sequence (x_n/n) is norm convergent to* $\sup x_n/n$.

PROOF. $y_n = x_n - x_1$ is a positive superadditive sequence. Again, $y_{2^k}/2^k$ is increasing and norm bounded, hence it is norm convergent to $z = \sup_k y_{2^k}/2^k = \sup_n y_n/n$. y_n/n is weak Cauchy and valued in $[0, z]$, hence it is weakly convergent to z. Lemma 1 applies and gives the norm convergence.

We can now precisely state our main result.

THEOREM 1. *For a countably order complete Banach lattice E the following properties are equivalent*:

(2.9) *E has an order continuous norm.*

(2.10) *For every positive, E-valued subadditive process (S_n) we have strong convergence of (S_n/n) with probability one.*

PROOF. Since (2.10) immediately implies (2.9) by Proposition 1, we focus on the more important implication of (2.10) by (2.9). If I denotes the invariant σ-field of θ, we first show

(2.11) $E^I[S_n/n]$ converges strongly to $Z = \inf_n E^I[S_n/n]$ for all ω.

Setting $Y_n = E^I[S_n]$ we have $Y_n \geqslant 0$ and

$$Y_{n+k} = E^I[S_{n+k}] \leqslant E^I[S_n + S_k \circ \theta^n] = Y_n + E^I[S_k \circ \theta^n] = Y_n + Y_k.$$

This proves for a.e. ω, $(Y_n(\omega))$ is a positive subadditive sequence in E so by Proposition 1 Y_n/n is norm convergent to $Z = \inf Y_n/n = \inf E^I[S_n/n]$ a.e.

Now since (S_n) is a sequence of Bochner integrable functions it is almost separably valued, so there is no loss in assuming E is separable. For $f \in E'_+$ we apply Kingman's theorem to the real process $X_n = f(S_n)$ to obtain

$$\lim \frac{1}{n} f(S_n) = \lim E^I\left(\frac{1}{n} f(S_n)\right) = \inf E^I\left(\frac{1}{n} f(S_n)\right) \tag{2.12}$$

outside a set Ω_f with $P(\Omega_f) = 0$. By (2.11), $E^I(S_n/n)$ converges strongly to $\inf E^I(S_n/n)$ so by the continuity of f we have

$$\lim f\left(E^I\left(\frac{S_n}{n}\right)\right) = f(\inf E^I(S_n/n)).$$

Since bounded linear functionals commute with conditional expectation, by (2.12) and the preceeding equality we have for $\omega \notin \Omega_f$ that

$$\begin{aligned} \lim f(S_n/n) &= \lim E^I\left(\frac{1}{n} f(S_n)\right). \\ &= \lim f(E^I(S_n/n) = f(\inf E^I(S_n/n)) = f(Z). \end{aligned} \tag{2.13}$$

We now adapt a technique used by Kingman [18] page 195, to show that S_n/n is majorized by a norm convergent sequence of random variables. For that, we fix $k > 0$ and suppose $n \geqslant k$. Letting $N = N(n)$ be the integral part of n/k, we see by subadditivity that

$$\begin{aligned} S_n &\leqslant \Sigma_{r=1}^N S_k \circ \theta^{(r-1)k} + S_{n-Nk} \circ \theta^{Nk} \\ &\leqslant \Sigma_{r=1}^N Z_r + W_N. \end{aligned} \tag{2.14}$$

Where $Z_r = S_k \circ \theta^{(r-1)k}$ and $W_N = \Sigma_{j=1}^{k-1} S_j \circ \theta^{Nk}$. Since W_N has the same distribution for all N and since $E\|W_1\| < \infty$, we have for all $\varepsilon > 0$ that

$$\Sigma_{N=1}^{\infty} P(\|W_N\| \geqslant \varepsilon N) = \Sigma_{N=1}^{\infty} P(\|W_1\| \geqslant \varepsilon N) \leqslant \frac{1}{\varepsilon} E\|W_1\| < \infty.$$

The Borel-Cantelli lemma thus implies that $\|W_N\|/N$ converges a.e. to zero. Since $\Sigma_{r=1}^N Z_r$ is an additive process, we have by Mourier's theorem that a.e.

$$\lim \frac{1}{N} \Sigma_{r=1}^N Z_r = E^I[Z_1] = E^I[S_k]. \tag{2.15}$$

By inequality (2.14) we have

$$Z \leq \frac{S_n}{n} \vee Z \leq \frac{1}{Nk}(\Sigma_{r=1}^{N} Z_r + W_N) \vee Z.$$

So by applying any f in E'_+ and letting n go to infinity we get for $\omega \notin \Omega_f$

$$F(Z) \leq \liminf f\left(\frac{S_n}{n} \vee Z\right) \leq \limsup f\left(\frac{S_n}{n} \vee Z\right) \leq f\left(E^I\left[\frac{S_k}{k}\right]\right).$$

Letting k tend to infinity we have

$$\lim f\left(\frac{S_n}{n} \vee Z\right) = f(Z) \qquad \text{outside} \quad \Omega_f.$$

Next in view of the identity $S_n/n \vee Z + S_n/n \wedge Z = S_n/n + Z$ and (2.13) we also have for $\omega \notin \Omega_f$ that

$$\lim f\left(\frac{S_n}{n} \wedge Z\right) = f(Z). \tag{2.16}$$

For every $\omega \in \Omega$, $(S_n/n)(\omega) \wedge Z(\omega)$ belongs to the weakly compact order interval $[0, Z(\omega)]$; hence there exists a subsequence (k_n) such that weak limit $S_{k_n}(\omega)/k_n \wedge Z(\omega)$ exists. Let $Z'(\omega)$ be such limit.
For every $f \in E'_+$, we have

$$\lim f\left(\frac{S_{k_n}(\omega)}{k_n} \wedge Z(\omega)\right) = f(Z'(\omega)) = f(Z(\omega)) \text{ if } \omega \notin \Omega_f,$$

so Z' is weakly measurable. Since E is separable, Z' is also strongly measurable. Moreover, we have $Z'(\omega) \leq Z(\omega)$ for each $\omega \in \Omega$, so it follows that Z' is Bochner integrable.

Now for any fixed $f \in E'_+$ we have for every $A \in \mathfrak{F}$ that $\lim \int_A f(S_n/n) \wedge Z) = \int_A f(Z') = \int_A f(Z)$. Since this last equality is valid for all $f \in E'$ we also have $\int_A Z' = \int_A Z$ and the validity of this equality for all $A \in \mathfrak{F}$ implies in turn that $Z = Z'$ a.s. We have thus proved that $(S_n/n) \wedge Z$ converges weakly a.s. to Z. By the first part of Lemma 1 we consequently have that $(S_n/n) \wedge Z$ converges strongly a.s. to Z. To apply this to the strong convergence of (S_n/n) we note

$$\begin{aligned}\left\|\frac{S_n}{n} - Z\right\| &\leq \left\|\frac{1}{Nk}(\Sigma_{r=1}^{N} Z_r + W_N) - \frac{S_n}{n}\right\| + \left\|Z - \frac{1}{Nk}(\Sigma_{r=1}^{N} Z_r + W_N)\right\| \\ &\leq \left\|\frac{1}{Nk}(\Sigma_{r=1}^{N} Z_r + W_N) - \frac{S_n}{n} \wedge Z\right\| + \left\|Z - \frac{1}{Nk}(\Sigma_{r=1}^{N} Z_r + W_N)\right\|\end{aligned}$$

where the last inequality follows from (2.14) and the fact that $Nk \leq n$. By the equality (2.15) we obtain

$$\limsup \left\|\frac{S_n}{n} - Z\right\| \leq \left\|E^I\left[\frac{S_k}{k}\right] - Z\right\| + \left\|Z - E^I\left[\frac{S_k}{k}\right]\right\|.$$

We already noted in (2.11) that $E^I[S_k/k]$ converges strongly a.s. to Z and this fact completes the proof.

As a corollary to Theorem 1 we note that the assumption of positivity can be weakened.

COROLLARY 1. *If E has an order continuous norm and δ is a Bochner integrable function such that for all $n > 1$*

$$\Sigma_{i=0}^{n-1}\delta \circ \theta^i \leqslant S_n$$

then S_n/n converges strongly with probability one.

PROOF. The corollary is immediate from the Mourier theorem and the fact that $S_n' = S_n - \Sigma_{i=0}^{n-1}\delta \circ \theta^i$ is a positive subadditive process.

The next theorem is the vector-valued analogue of Kingman's ergodic decomposition of a real subadditive process.

THEOREM 2. *If E is weakly sequentially complete and (S_n) is a subadditive process such that*

$$\sup \frac{1}{m} E\|\Sigma_{i=1}^{m}(S_i - S_{i-1} \circ \theta)\| < \infty \tag{2.17}$$

then there is a Bochner integrable random variable δ such that for $n > 0$ we have

$$S_n \geqslant \Sigma_{i=1}^{n-1}\delta \circ \theta^i \text{ and } E(\delta) = \inf \frac{1}{m} E(S_m).$$

PROOF. We let $S_n' = \Sigma_{i=0}^{n-1}S_1 \circ \theta^i - S_n$ and note that by the subadditivity of S_n that S_n' is *positive* and superadditive. Also, since $S_i' - S_{i-1}' \circ \theta = S_{i-1} \circ \theta - S_i + S_1$ we have

$$\|\Sigma_{i=1}^{m}(S_i' - S_{i-1}' \circ \theta)\| \leqslant \|\Sigma_{i=1}^{m}(S_{i-1} \circ \theta - S_i)\| + m\|S_1\|$$

so (2.17) yields

$$\sup \frac{1}{m} E\|\Sigma_{i=1}^{m}(S_i' - S_{i-1}' \circ \theta)\| < \infty. \tag{2.18}$$

We will use S_n' to construct a δ' such that

$$S_n' \leqslant \Sigma_{i=0}^{n-1}\delta' \circ \theta^i \text{ for all } n > 0$$

and such that $E(\delta') = \sup (1/n)\, E(S_n')$. By the correspondence between S_n and S_n' the δ required by the theorem is then given by $\delta = S_1 - \delta'$.

We first note by (2.18) that

$$\sup \frac{1}{n}\|E(S_n')\| = \sup \frac{1}{n}\|E[\Sigma_{i=1}^{n}(S_i' - S_{i-1}' \circ \theta)]\|$$

$$\leqslant \sup \frac{1}{n} E\|\Sigma_{i=1}^{n}(S_i' - S_{i-1}' \circ \theta)\| < \infty$$

so Proposition 2 implies that the superadditivity of $E(S_n')$ entails the convergence of $(1/n)E(S_n')$ to $\sup(1/n)E(S_n')$.

As before we may assume without loss that E is separable. Consequently, E contains a quasiinterior point u (Schaefer [20] page 97). We will use this quasiinterior point to push through a truncation argument parallel to one used in the real case in Ackoglu-Sucheston [1].

We let $\phi_m = (1/m)\Sigma_{i=1}^{m}(S_i' - S_{i-1}' \circ \theta)$ and note that the same computation as in the real case of Kingman's lemma will give

$$\Sigma_{i=0}^{n-1}\phi_m \circ \theta^i \geq \left(1 - \frac{n-1}{m}\right)S_n' \text{ for } 1 \leq n < m. \tag{2.19}$$

Now for $j \geq 0$ the sequence $(\phi_m \wedge ju)_m$ is in the weakly compact interval $[0, ju]$ of $L^1[E]$ so by diagonalization procedure there are $\lambda_j \in L^1[E]$ which are weak limits in $L^1[E]$ of $(\phi_{m_k} \wedge ju)$ for a fixed subsequence (m_k) and all j. By the weak lower-semicontinuity of the norm in $L^1[E]$ we have

$$\liminf_{m_k} E\|\phi_{m_k} \wedge ju\| \geq E\|\lambda_j\|. \tag{2.20}$$

We also have

$$\sup_j E\left[\|\lambda_j\|\right] \leq \sup_{m_k} E\|\phi_{m_k}\| < \infty$$

thus the (λ_j) are increasing and norm bounded so by the weak sequential completeness of E the (λ_j) converge to their supremum δ'.

By (2.19) we have

$$\begin{aligned}\Sigma_{i=0}^{n-1}(\phi_{m_k} \wedge ju) \circ \theta^i = \left(\Sigma_{i=0}^{n-1}\left(\phi_{m_k} \circ \theta^i\right) \wedge ju\right) &\geq \left(\Sigma_{i=0}^{n-1}\phi_{m_k} \circ \theta^i\right) \wedge ju \\ &\geq \left(1 - \frac{n-1}{m_k}\right)S_n' \wedge ju\end{aligned}$$

so taking weak limits with $m_k \to \infty$ we have

$$\Sigma_{i=0}^{n-1}\lambda_j \circ \theta^i \geq S_n' \wedge ju.$$

Now let $j \to \infty$ to obtain the key relation

$$\Sigma_{i=0}^{n-1}\delta' \circ \theta^i \geq S_n' \text{ for all } n > 0.$$

Taking expectations further yields

$$E\delta' \geq \sup \frac{1}{n}E\left[S_n'\right]$$

and to get the opposite inequality it is enough to notice that

$$E[\delta] = \sup E\left[\lambda_j\right] \leq \sup E\left[\phi_m\right] = \sup \frac{1}{n}E\left[S_n'\right]$$

so the proof is complete.

COROLLARY 2. *For a Banach lattice E the following are equivalent*:

(2.21) *E is weakly sequentially complete.*

(2.22) *For every E-valued subadditive process* (S_n) *such that*

$\sup_n 1/n\|\Sigma_{i=1}^n(S_i - S_{i-1}\circ\theta)\| < \infty$ *we have*

S_n/n *converges in norm* a.e.

PROOF. That (2.21) implies (2.22) follows from the theorem above and Corollary 1. For the converse it suffices to invoke Proposition 2.

For a real subadditive process the finiteness of the time constant $\inf(1/n)E(S_n)$ is sufficient for the a.s. convergence of S_n/n. The next result shows that the spaces for which a corresponding result holds are precisely those isomorphic to $L^1[\mu]$ for some μ, (i.e., A L space).

THEOREM 3. *For a Banach lattice E, the following are equivalent*:

(2.23) *E is isomorphic (as a Banach lattice) to an* A L *space.*

(2.24) *For every E-valued subadditive process* (S_n) *such that*

$\inf(1/n)E(S_n)$ *exists, we have* S_n/n *converges in norm* a.s.

PROOF. By the same correspondence used in Theorem 2 we can work with positive superadditive processes (S_n) such that $\sup_n(1/n)E[S_n]$ exists. That (2.24) follows from (2.23) is a consequence of the fact that an A L space is weakly sequentially complete and that there exists an $f \in E'_+$ such that $\|x\| \leqslant f(|x|)$ for all $x \in E$. To use this second fact, we note

$$\sup \frac{1}{n}E\|\Sigma_{i=1}^n(S_i - S_{i-1}\circ\theta)\| \leqslant \sup_n\frac{1}{n}E\Sigma_{i=1}^n f(S_i - S_{i-1}\circ\theta)$$

$$= \sup_n\frac{1}{n}E(f(S_n)) = f\left(\sup\frac{1}{n}E(S_n)\right) < \infty.$$

To show the opposite implication we use the characterization of A L which says a space is A L if and only if every positive summable sequence is absolutely summable (see Schaefer [21] page 242). To that end, suppose there is a sequence (x_n) in E_+ such that $\Sigma_{n\geqslant 0}x_n = x$ and $\Sigma_{n\geqslant 0}\|x_n\| = \infty$. We can select an increasing sequence of integers (m_k) such that $\Sigma_{i=m_k+1}^{m_{k+1}}\|x_i\| \geqslant 1$ and by multiplying by coefficients less than 1, we may assume the sum equals 1.

For every $k \geqslant 1$ divide the interval [0, 1] into $m_{k+1} - m_k$ subintervals $(A_{i,k})_{i=m_{k+1}}^{m_{k+1}}$ such that the length of $A_{i,k}$ is $\|x_i\|$. Now we can define an E-valued process (S_n) by

$$S_n = n\Sigma_{k=1}^n\Sigma_{i=m_k+1}^{m_{k+1}}\frac{x_i}{\|x_i\|}1_{A_{i,k}}. \tag{2.25}$$

If we take θ to be the identity transformation on [0, 1], one can easily check that S_n

is a superadditive process. Now

$$\frac{1}{n}E(S_n) = \Sigma_{k=1}^{n} E \Sigma_{i=m_k+1}^{m_{k+1}} \frac{x_i}{\|x_i\|} 1_{A_{i,k}} = \Sigma_{k=1}^{n} \Sigma_{i=m_k+1}^{m_{k+1}} x_i \leqslant x.$$

On the other hand for any $\omega \in [0, 1]$ we have for each k an i_k such that $\omega \in A_{i_k, k}$, and, consequently,

$$\left\| \frac{1}{n} S_n(\omega) \right\| = \left\| \Sigma_{k=1}^{n} \frac{x_{i_k}}{\|x_{i_k}\|} \right\|. \tag{2.26}$$

Since each of the summands is positive and of norm 1, the expression given in (2.26) tends to infinity with n, and the proof of Theorem 3 is complete.

3. Examples and applications. The usefulness of subadditive ergodic theory is substantially extended by a reformulation in terms of stationary processes. After sketching this reformulation we will indicate several historically fecund sources of subadditive processes and analyze a problem from the theory of empirical distributions.

We will call a doubly indexed family of Bochner integrable random variables (Y_{st}) $s, t \in Z_+$ a *subadditive process* provided

(3.1) $\qquad Y_{su} \leqslant Y_{st} + Y_{tu}$ whenever $s < t < u$

(3.2) $\qquad (Y_{st})$ has the same joint distributions as $(Y_{s+1,t+1})$.

To unite this definition with the one given in Section 1 we note first that (Y_{st}) can be extended to a process which satisfies (3.1) and (3.2) for all $s, t \in Z$ not just Z_+. Next Ω can be taken to be the set of functions

$$x\colon \{(s, t); s, t \in Z, s < t\} \to E$$

which satisfy

$$x(s, u) \leqslant x(s, t) + x(t, u), \qquad (s < t < u).$$

The measure P on Ω is the measure induced by the process satisfying (3.1) and (3.2), and θ is defined as the shift

$$(\theta x)(s, t) = x(s+1, t+1).$$

This correspondence is just the E analogue of the one introduced by Kingman, and for further details one can consult Kingman [18] pages 181 and 186.

For unity of this section on applications we restate our main result in terms of our second definition of subadditive processes.

THEOREM 1. *For a Banach lattice E with order continuous norm, and for every positive process (Y_{st}) satisfying* (3.1) *and* (3.2) *we have Y_{0n}/n converges in norm with probability one.*

This reformulation makes available a number of recipes for producing subadditive processes, some of which we now consider.

Unconstrained maximization. This is perhaps the most natural source of subadditive processes, and the theory of random series (as studied in Kahane [15]) suggests many specific examples. To take one, suppose X_i $i = 1, 2, \cdots$ are i.i.d. Bochner integrable random variables valued in a Banach lattice E. We let

$$Y_{s,t} = \max_{\varepsilon_i = \pm 1} |\varepsilon_{s+1} X_{s+1} + \varepsilon_{s+2} X_{s+2} + \cdots + \varepsilon_t X_t|$$

where the maximum is taken in the sense of the lattice E and extends over all choices of the signs ε_i. Even in the simple case $E = L^2[0, 2\pi]$ and $X_i = \cos(x + Z_i)$ where Z_i are i.i.d. rv's, it is useful to be able to conclude from Theorem 1 that $n^{-1} Y_{0n}$ converges strongly with probability one.

Constrained minimization. This is a source of subadditive processes which goes back to the pioneering work of Hammersley and Welsh [14]. Suppose for each edge e of a connected graph G we associate a positive integrable vector valued random variable $X(e)$. Given vertices v, v' of G we define

$$U(v, v') = \inf_P \Sigma_{e \in P} X(e)$$

where the infimum is taken over all paths P from v to v' in G and the sum is over all edges in P. When G is $Z \times Z$ with edge set given by joining a point to its four neighbors, we can then define a subadditive process by

$$Y_{s,t} = U((s, 0), (t, 0)).$$

A particularly interesting choice of $X(e)$ is given by taking independent realizations of the absolute value of the Brownian bridge and where $E = L^1[0, 1]$. In connection with this specific choice it would be interesting to determine the value of the time constant

$$\lim_{t \to \infty} t^{-1} E(Y_{0,t}) = \gamma,$$

but this problem remains open.

Subadditive operators on random walks. Suppose (X_i) are i.i.d. positive E_1-valued random variables and that M is a positive subadditive operator from E_1 to a Banach lattice E_2. The process

$$Y_{s,t} = M(X_{s+1} + X_{s+2} + \cdots + X_t)$$

will then be a subadditive process in E_2. The operator M required above can be taken to be the Hardy-Littlewood maximal operator on L^P or any of the similar operators described by Stein [24]. Although the continuity of M will often allow one to side step Theorem 1, the use of subadditive processes retains a conceptual value and may prove valuable in more general contexts.

Beyond general sources of subadditive processes sketched above one can gain repeated inspiration from the examples given in Hammersley and Welsh [13], Hammersley [12], and Kingman [16, 17, 18].

We now turn to a more detailed discussion of a problem from the theory of empirical distributions which can be solved by means of vector valued subadditive processes and the application of Theorem 1.

Uniformity classes and empirical distributions. We now assume (X_i) is a sequence of i.i.d. random variables valued in R^d with distribution μ. A well-known and much studied problem is the following:

Under which conditions on a class $\mathcal{C}$ of Borel subsets of R^d does one have

$$\sup_{A \in C} \left| \frac{1}{n} \Sigma_{i=1}^n 1_A(X_i) - \mu(A) \right| \to 0 \text{ a.s. as } n \to \infty?$$

Many particular classes $\mathcal{C}$ have been studied and a number of these are invariant under the group G of rigid motions of R^d. For example, the class of convex sets are studied in Rao [20], the half-spaces in Wolfowitz [27], or many of the general classes studied in Vapnik and Chervonenkis [26], Steele [22], Dudley [9] or Dudley and Kuelbs [10].

There are also several classes of interest which are not invariant under G. Here one should consider the class of rectangles, class of lower layers [4, 23] or any class containing only a countable number of elements. Now for any class $\mathcal{C}$ we can define a new class $\mathcal{C}_g = \{B\colon B = gA, A \in \mathcal{C}\}$ by translating the elements of $\mathcal{C}$. Similarly we have the translated discrepancy

$$\phi_n(g; \omega) = \sup_{A \in C_g} \left| \frac{1}{n} \Sigma_{i=1}^n 1_A(X_i) - \mu(A) \right|.$$

A class $\mathcal{C}$ for which $\phi_n(e; \omega) \to 0$ a.s. for the identity e is called a *uniformity class*, and the class formed by the union of all translates of $\mathcal{C}$ is called the closure of $\mathcal{C}$ under rigid motions (i.e., $\bar{\mathcal{C}} = \cup_{g \in G} \mathcal{C}_g$).

It is natural to ask if $\bar{\mathcal{C}}$ is a uniformity class provided $\mathcal{C}$ is a uniformity class. Unfortunately one can construct even a singleton class $\mathcal{C} = \{A\}$ such that $\bar{\mathcal{C}}$ is not a uniformity class. (To prove this one can use the construction given in Steele [24] for a related problem.)

Even though $\bar{\mathcal{C}}$ need not be a uniformity class, the natural measure on G (right Haar measure) shows that $\bar{\mathcal{C}}$ shares some properties with $\mathcal{C}$.

THEOREM 4. *If $\mathcal{C}$ is a class of Borel subsets of the unit ball of R^d, and the functions*

$$\phi_n(g; \omega) = \sup_{A \subset \mathcal{C}_g} \left| \frac{1}{n} \Sigma_{i=1}^n 1_A(X_i) - \mu(A) \right|$$

are measurable then they converge in $L^1(G)$ with probability one.

PROOF. We first consider the process

$$Y_{s,t} = \sup_{A \in \mathcal{C}} \left| \Sigma_{i=s+1}^t 1_{g(A)}(X_i) - (t-s)\mu(g(A)) \right|.$$

To check that $Y_{s,t}$ is valued in $L^1(G)$ we note

$$0 \leqslant Y_{s,t} \leqslant \Sigma_{i=s+1}^t 1_{g(A)}(X_i) + (t-s)\mu(g(A)).$$

Since $1_{g(B)}(x)$ is 0 except on a compact subset of G we have $\Sigma_{i=g+1}^{t} 1_{g(B)}(X_i) \in L^1(G)$. For the second term we note,

$$\int_G \mu(g(A))\, dm(g) = \int_G \int_{R^d} 1_{g(A)}(x)\, d\mu(x)\, dm(g)$$
$$= \int_{R^d} \left[\int_G 1_{g(A)}(x)\, dm(g) \right] d\mu(x) = \int_{R^d} \left[\int_G 1_{g(A)}(0)\, dm(g) \right] d\mu(x) < \infty$$

since μ is a probability measure and since the last integrand does not depend on x. The remaining properties which show $Y_{s,t}$ is an $L^1(G)$ valued, positive, subadditive process are also easily checked. Hence by Theorem 1, we see $\phi_n(g) = (1/n)Y_{0,n}$ converges strongly in $L^1(G)$ with probability one (i.e., there is a $\phi \in L^1(L^1(G))$ such that $\int |\phi_n(g, \omega) - \phi(g, \omega)|\, dm(g) \to 0$ for a.e. ω).

PROPOSITION 3. *If in addition to the hypotheses of Theorem* 4 *we assume* $\mathcal{C}_g$ *is a uniformity class for each* $g \in G$ *then* ϕ_n *converges strongly to* 0 *with probability one.*

PROOF. Let ϕ be the limit of the ϕ_n guaranteed by Theorem 4. For any fixed g we have a.s. that

$$\phi(g, \omega) = \lim_{n\to\infty} \phi_n(g, \omega) = 0,$$

hence

$$0 = \int\int_{G\Omega} \phi(g, \omega)\, dP\, dm(g) = \int_\Omega \left[\int_G \phi(g, \omega)\, dm(g) \right] dP.$$

Since $\phi(g, \omega)$ is positive this identity says $\phi(g, \omega)$ is the zero element of $L^1(G)$ for a.e. ω.

Theorem 4 and Proposition 3 have easy extensions to $L^p(G)$ for $1 \leqslant p < \infty$ which we omit. As we commented before, there is a singleton class $\mathcal{C} = \{A\}$ for which ϕ_n does not converge to 0 in $L^\infty(G)$, even though we now see it must converge to 0 in $L^p(G)$ for all $1 \leqslant p < \infty$. This example is particularly useful to keep in mind considering the assumption of Theorem 1 on E that E have an order continuous norm (equivalently, $l^\infty \not\hookrightarrow E$).

REFERENCES

[1] AKCOGLU, M. A. and SUCHESTON, L. (1978). A ratio ergodic theorem for superadditive processes. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **44** 269–278.

[2] BURKHOLDER, D. L. (1968). Contribution to the discussion of [16].

[3] CARTWRIGHT, D. (1974). The order completeness of some spaces of vector valued functions. *Bull. Austral. Math. Soc.* **11** 57–61.

[4] DEHARDT, J. (1971). Generalization of the Glivenko-Cantelli theorem. *Ann. Math. Statist.* **42** 2050–2055.

[5] DEL JUNCO, A. (1977). On the decomposition of a subadditive stochastic process. *Ann. Probability* **5** 298–302.

[6] DERRIENNIC, Y. (1975). Sur le theorem ergodique sous-additif. *C. R. Acad. Sci. Paris Ser. A* **281** 985–988.

[7] DIESTEL, J. and SEIFERT, C. J. (1976). An averaging property of the range of a vector measure. *Bull. Amer. Math. Soc.* **82** 907–909.

[8] DIESTEL, J. and UHL, J. J. (1977). *Vector Measures*. Mathematical Surveys AMS **15**.

[9] DUDLEY, R. (1978). Central limit theorems for empirical measures. *Ann. Probability* **6** 899–929.

[10] DUDLEY, R. and KUELBS, J. (1978). Log log laws for empirical measures. Unpublished manuscript.

[11] GARLING, D. H. (1978). Sums of Banach space valued random variables. *Lecture Notes*, Springer, New York. To appear.
[12] HAMMERSLEY, J. M. (1972). A few seedlings of research. *Proc. Sixth Berkeley Symp. Math. Statist. Probability* **1** 345–394. Univ. California Press.
[13] HAMMERSLEY, J. M. and WELSH, J. A. D. (1965). First passage percolation, subadditive processes, stochastic networks, and generalized renewal theory. In *Bernouilli-Bayes-Laplace Anniversary Volume*. Springer, Berlin.
[14] HEINICH, H. (1978). Convergence of positive submartingales in a Banach lattice. *C. R. Acad. Sci. (Paris), Sec. A* **286** 279–280.
[15] KAHANE, J-P. (1968). *Some Random Series of Functions*. D. C. Heath, Lexington.
[16] KINGMAN, J. F. C. (1968). The ergodic theory of subadditive stochastic processes. *J. Roy. Statist. Soc. Ser. B* **30** 499–510.
[17] KINGMAN, J. F. C. (1973). Subadditive ergodic theory. *Ann. Probability* **1** 883–905.
[18] KINGMAN, J. F. C. (1976). Subadditive processes. *Ecole d'Été de Probabilités de Saint-Flour*. Lecture Notes in Math. **539** 168–223. Springer, Berlin.
[19] MOURIER, E. (1953). Elements aleatoires dans un espace de Banach. *Ann. Inst. H. Poincaré Sect. B* **13** 161–244.
[20] RAO, R. RANGA (1962). Relations between weak and uniform convergence of measures with applications. *Ann. Math. Statist.* **33** 659–680.
[21] SCHAEFER, H. H. (1975). *Banach Lattices and Positive Operators*. Springer, New York.
[22] SMELTZER, M. D. (1977). Subadditive stochastic processes. *Bull. Amer. Math. Soc.* **83** 1054–1055.
[23] STEELE, J. M. (1978a). Empirical discrepancies and subadditive processes. *Ann. Probability* **6** 118–127.
[24] STEELE, J. M. (1978b). Covering finite sets by ergodic images. *Canad. Math. Bull.* **21** 85–92.
[25] STEIN, E. M. (1970). *Singular Integrals and Differentiability Properties of Functions*. Princeton Univ. Press.
[26] VAPNIK, V. N. and CHERVONENKIS, A. YA (1971). On the uniform convergence of relative frequencies of events to their probabilities. *Theor. Probability Appl.* **16** 264–280.
[27] WOLFOWITZ, J. (1960). Convergence of the empiric distribution function on half-spaces. In *Contributions Probability and Statistics*. Stanford Univ. Press.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF BRITISH COLUMBIA
VANCOUVER, B.C.
CANADA

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA