Modeling the Yield Curve

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Plan for Talk

Background
- What is the yield curve?
- What makes it interesting and important?

Examples
- Cash
- Commodities (primarily crude oil)

Data analysis is simpler with prices
- How to recognize arbitrage?
- More natural structure

Commodities are different
- Linkages among products
Background
What is the yield curve?

- Interest rates earned on treasury bills/bonds of different maturity (7 Nov 08).
- Curve on a given day is formed by connecting the rates over different maturities.

<table>
<thead>
<tr>
<th>Term</th>
<th>Interest</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 month</td>
<td>0.269%</td>
</tr>
<tr>
<td>6 month</td>
<td>0.84%</td>
</tr>
<tr>
<td>2 year</td>
<td>1.32%</td>
</tr>
<tr>
<td>5 year</td>
<td>2.56%</td>
</tr>
<tr>
<td>10 year</td>
<td>3.79%</td>
</tr>
<tr>
<td>30 year</td>
<td>4.27%</td>
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</tbody>
</table>
What is the yield curve?

- Interest rates derived from contracts in the Eurodollar option market.
- London Interbank Offered Rate (LIBOR)
- Interest rate on a 3-month contract in the future
- Plot shows instantaneous forward rates.

Many more contracts
Yields for Commodities

- Interest rates implied by the prices of contracts for delivery of a commodity at some future date.

- Light crude oil, same date as prior slide
Questions

- What are the dynamics of the yield curve?
- How fast does it change? Can you predict where it’s headed?
- How is the yield for cash related to yields implied by commodities that include convenience factors?
- How are yields for various products related to one another?
- What’s the connection between these “curves” and the underlying data?
Plots: Cash

- Yields on cash over period of about 100 days
- Red curve is the original
- Gray curves are separated by 10 trading days
- Some changes are large, other curves cluster together
Plots: Light Crude

- Yields on crude over same 100 days
- Red curve is the original, gray 10 days apart
- Rather different appearance from cash, with much less smoothness

Surface plots are also entertaining
Relations Among Products

- Dependent movements over 10-day intervals for heating oil and light crude.
- Strong seasonal pattern for heating oil that is not apparent in yield on light crude.
Models

- Models provide parsimonious way to predict where the curve is heading.
- Rather than have to predict a “curve”, forecast the value of certain parameters in a regression-like formula for the curve.

- Model used resembles a polynomial, but on a logarithmic scale more suited to description of rates.

- Modeling issues (see paper)
  - How many polynomial terms?
  - Does the model allow arbitrage?
Decomposition

Decompose the yield curve $y_t(T)$ into three components

$$y_t(T) = U(T) + M_t(T) + D_t(T)$$

Long-term unconditional expectation

$$E_s y_s(T) = U(T)$$

Other terms are separable in $T$ and $t$, factoring as $M_t(T) = m_t g(T)$

Maturity specific component is mean reverting

$m(t)$ follows log normal SDE with expectation

$$E_s m_t = m_s e^{-k(t-s)} \quad s < t$$
Decomposition, cntd

- Date specific term is also mean reverting, but captures effects that move toward origin with time

\[ E_s D_t(T) = D_s(T+t-s) \quad s < t, T \]

Each contract carries the date-specific effect

- If \( d_t \) follows log normal SDE, then \( h(T) = e^{-kT} \)

\[ E_s D_t(T) = (d_s e^{-k(t-s)}) e^{-kT} = D_s(T+t-s) \]

- Example \((2,2,3)\)
  - 2\(^{nd}\) order unconditional curve
  - 2 maturity-specific functions
  - \( m_{t,1} = c_{m1} (e^{-kT} - e^{-2kT}) \), \( m_{t,2} = c_{m2} (e^{-kT} - e^{-4kT}) \)
  - 3 date-specific functions
Component Functions

- Basis elements for expressing a model for the yield curve using component SDEs
- Constant $k$ determines shapes

$M_T(\tau)$

$D_T(\tau)$
Fitted Yield Curves

- Extracted state coefficients for several days
- Estimates are smoother than those for each day
- Unconditional, maturity, date components
- Observation error ought to be uncorrelated
Forecasting

- Extrapolate fit
  - Recursively update as forecast extends beyond initial training period
  - Negative values indicate our models dominate random walk (Newey-West statistic)

<table>
<thead>
<tr>
<th></th>
<th>65 days</th>
<th>125 days</th>
<th>250 days</th>
</tr>
</thead>
<tbody>
<tr>
<td>Crude</td>
<td>2.1</td>
<td>-0.09</td>
<td>-2.2</td>
</tr>
<tr>
<td>Heating Oil</td>
<td>-5.8</td>
<td>-8.7</td>
<td>-2.2</td>
</tr>
<tr>
<td>Natural Gas</td>
<td>-1.6</td>
<td>-2.2</td>
<td>-1.4</td>
</tr>
<tr>
<td>Soybean</td>
<td>-5.1</td>
<td>-7.4</td>
<td>-3.1</td>
</tr>
</tbody>
</table>
Never underestimate the time that it takes to prepare the data.
Prices

- Yields are not directly observed
- Obtain prices of futures contracts from data vendor CRB Trader
- Prices are much smoother
From Price to Yield

- Compute the yield at the midpoint between two observed prices as the continuously compounded rate of interest:
  \[ y_t(\tau) = \frac{\log p_{t}(\tau+d) - \log p_{t}(\tau)}{d} \]  as \( d \to 0 \)

Results
- Differencing magnifies random noise
- Observed only at discrete set of points
- Terminal date is about 15 days
- No observed spot rate
- Anomalies (aka, “market microstructure”)
Anomaly in Cash Yields

- Yield curves are about 10 days apart
- Ripple induced by preference for contract at a specific date (market microstructure)
Anomaly: Sticky Price

- A contract for light crude did not trade this day, so price stayed same as on prior day
- Rest of the curve shifted
Patching Anomalies

- Adaptive procedure that will not introduce side effects.
- Approach is to follow a contract over time rather than fixed maturity.
- Avoid interpolation-induced transitions.
- Contracts have a more consistent sequential pattern than in yields over varying maturities.
- Contracts follow a diagonal in the data $y_{t}(T)$ for fixed expiry date $t+T$.
Fixing the Anomalies

- Follow a contract over time
- Outliers are more easily identified in the sequence of yields associated with contract rather than over the yield curve itself.

Which are real, which are flukes?
Does not remove all...

In order not to distort “real” prices, miss a few outliers.
Nor a sustained problem...

Moving medians of length 3 cannot patch a period with a sustained anomaly.

persists for about 3 weeks
Contracts
Time Series Models

- Structural model
  - Describes the yield curve as a function $f(T)$
  - Can evaluate $f(T)$ at any maturity
  - Theoretical properties, derived quantities

- Time series
  - Considerable methodology available
  - Cannot hold maturity $T$ fixed unless were able to observe $y_T(T)$ for all $t$
    - Interpolation between points introduces artifacts
  - Following a single contract provides most direct series
Yields of Contracts

Think of data as collection of contracts $y_t(T)$ identified by expiry date

Contracts end, becoming more variable as $T$ approaches 0 where yield is more volatile.
SD of Contracts

- Standard deviation of contracts as $\tau$ approaches zero.
- Very strong presence of outliers as $\tau$ nears 0.
Cointegration

- Differences in yields of adjacent contracts are non-stationary as time gap increases.
- Suggests that contracts are cointegrated when not “too far” apart.
- Degree of time separation also measures the rate of change in the yield curve, a sort of stochastic modulus of continuity of the yield.
- Interpret the source of non-stationarity as due to movements of some underlying yield process.
- Latent variable type of model.
Prices of Contracts

- Prices again seem simpler to use
- Plot shows prices of 4 contracts for light crude, about 100 days apart

mid 1984
Prices of Contracts

- Prices again seem simpler to use
- Plot shows prices of 4 contracts for light crude, about 100 days apart
More things

- Multivariate structure
  - How to use the evident contemporaneous movements in prices or yields for various products (such as the various soy commodities)
  - Related to movements in yield for cash as well
- Model after subtracting out the yield for cash to extract a convenience yield
- Implications for stationarity after remove the yield: simpler model?
Summary

- Models for yield curve
  - Capture dynamics of yields
  - Date and maturity specific decomposition relevant for commodities as well
  - Out-of-sample performance superior to random walk benchmark

- Data analysis suggests reasons to work with contracts rather than a function to maturity
  - Outliers
    - More amenable to statistical methods

- Many unresolved questions