Valuing Investments
A Statistical Perspective

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Challenge

Value of $1 initially invested in 1980 and reinvested

Convincing going forward?
A Special Opportunity!

- While you are thinking about those dice, here's a special opportunity...

The Bob Fund

- Guarantees 2% excess annual returns above any benchmark you want. Guaranteed.
- Rest assured, it's not a Ponzi/Madoff scheme.
- Contact me after the talk...
Overview

- **Principles**
  - Focus on returns, not cumulative value or prices
    - One good year ≠ continuing success
  - Remove market performance (CAPM)
    - Leverage was good until lately
  - Watch for unseen volatility (Peso problem)
    - Just because it has not happened does not make it impossible.
  - Adjust for multiplicity
    - Happy to see resurgence of E-trade ads

- **How to evaluate investments ...**
  - As random process
  - Offerings of financial advisors
  - Using data
Returns
Overall Market Performance


7% Annual growth

Log Scale

Data: Yahoo Finance, Jan 1950 - Feb 2009, 710 months
Cumulative Returns?

Too easy to be deceived...

Moral: Stick to returns...
Monthly Returns

Much simpler structure, almost iid...

\[
\frac{P_t - P_{t-1}}{P_{t-1}}
\]

October 1987
“Black Monday”

August 1998
Long Term Capital Banking Crisis

October 2008
Distribution of Returns

\( \text{mean} = 0.0064, \ s = 0.0415, \ s^2 = 0.0017 \)

Fat tails more apparent in daily data.

\( \text{mean} = 0.0003, \ s = 0.0095, \ s^2 = 0.0001 \)
Challenge Asset Returns

Sequence of “bets” that appear nearly independent, but correlation remains between assets.

Risk = variation in returns

\[ R_t = \frac{P_t - P_{t-1}}{P_{t-1}} \]
## Summary Statistics

### Monthly returns, 1980–2009

<table>
<thead>
<tr>
<th></th>
<th>Mean Return</th>
<th>SD Return</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exxon</td>
<td>0.0089</td>
<td>0.0503</td>
</tr>
<tr>
<td>Berkshire</td>
<td>0.0137</td>
<td>0.0701</td>
</tr>
<tr>
<td>Bob</td>
<td>0.0064</td>
<td>0.0467</td>
</tr>
<tr>
<td>TEAM</td>
<td>0.0014</td>
<td>0.0217</td>
</tr>
</tbody>
</table>

Trade off return for risk?
Questions

- Two fundamental questions
- How much?
  - How much of my wealth should I invest to meet my financial goals?
- Which assets?
  - Start with the whole-market index
  - Which other investments in addition to an index based on the whole market?
How Much?
The Dice Game
What makes a good investment?

Consider 3 investments...

<table>
<thead>
<tr>
<th>Investment</th>
<th>Average Annual Return</th>
<th>SD Annual Return</th>
</tr>
</thead>
<tbody>
<tr>
<td>Green</td>
<td>7.5%</td>
<td>20%</td>
</tr>
<tr>
<td>Red</td>
<td>71%</td>
<td>132%</td>
</tr>
<tr>
<td>White</td>
<td>0%</td>
<td>6%</td>
</tr>
</tbody>
</table>

Questions

Which of these do you like, if any?
How do you decide: risk versus return?
Hands-on Simulation

3 dice determine outcomes:

\[ W_t = (\text{Table Result}) \cdot W_{t-1} \]

<table>
<thead>
<tr>
<th>Outcome</th>
<th>Green</th>
<th>Red</th>
<th>White</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.8</td>
<td>0.06</td>
<td>0.9</td>
</tr>
<tr>
<td>2</td>
<td>0.9</td>
<td>0.2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1.05</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>1.1</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>1.2</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>1.4</td>
<td>3</td>
<td>1.1</td>
</tr>
</tbody>
</table>

"Being Warren Buffett", Amer Statistician, 2006
Typical Results

- Red is “exciting” but generally loses value.
- Green offers steady growth.
- White goes nowhere.

Green is calibrated to match annual excess returns on US stock market.

White is calibrated to match returns on Treasury Bills.

We made up Red!
Occasional Results

- Red soars...
- In 20 rounds, the expected value of Red is $1.71^{20} = 45,700$ times initial value
Digesting the Results

- Something to ponder
  - Most simulations with the dice result in Red having lost most of its value.
  - A few simulations end with Red being fabulously wealthy, the “Warren Buffetts” of the class

- In the long run, Red will lose (w.p. 1)
  - How can I recognize that Red will lose without waiting for it to happen?
  - Even so, how can I take advantage of Red?
# Investment Objective

- **Long-run wealth**
  \[ W_t = W_{t-1} (1+r_t) \]
  \[ = W_0 (1+r_1)(1+r_2) \ldots (1+r_t) \]

- If the \( r_t \) are independent over time, then
  \[ W_t \approx W_0 (1+E(r_t) - \text{Var}(r_t)/2) \]

## Volatility Drag

<table>
<thead>
<tr>
<th></th>
<th>( E(r_t) )</th>
<th>( \text{Var}(r_t) )</th>
<th>( E(r_t) - \text{Var}(r_t)/2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Green</td>
<td>0.075</td>
<td>( (0.20)^2 = 0.04 )</td>
<td>( .075 - .04/2 = .055 )</td>
</tr>
<tr>
<td>Red</td>
<td>0.71</td>
<td>( (1.32)^2 = 1.74 )</td>
<td>-0.16</td>
</tr>
<tr>
<td>White</td>
<td>0</td>
<td>( (0.06)^2 = 0.003 )</td>
<td>-0.002</td>
</tr>
</tbody>
</table>

*Can buy this one*
Diversifying is good.

- Mix investments rather than leaving everything in one.
- Pink is a 50/50 mixture of Red & White.
  \[
  E(\text{Pink}) = E(0.5 \text{ Red} + 0.5 \text{ White}) \\
  = E(\text{Red})/2 = 0.355 \\
  \text{Var}(\text{Pink}) = \text{Var}(0.5 \text{ Red} + 0.5 \text{ White}) \\
  = \text{Var}(\text{Red})/4 = 0.435
  \]
- Long-run value of Pink is positive:
  \[
  E(\text{Pink}) - \text{Var}(\text{Pink})/2 = 0.14
  \]
even though neither Red nor White perform well taken separately.

Sacrifice half of the return to reduce the variance by 4.
Lessons from Dice Game

- Long-run return given by
  \[ E(\text{return}) - \frac{1}{2} \text{Var}(\text{return}) \]

- Over short horizons, a poor long-term investment might appear very attractive.

- Portfolios succeed by trading expected returns for reductions in variance
Cautions

- Real investments lack some properties of the investments in the dice simulation

- Independence
  - The dice fluctuate independently of one another. The returns of Red are not affected by what happens to Green.

- Stability
  - The properties of the dice stay the same throughout the simulation. The chance for a good return on Red does not change.

- Parameters known
  - We know the properties of the random processes in the dice game.
# Challenge: Summary Stats

Monthly returns, 1980–2009

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<td>0.0089</td>
<td>0.0503</td>
<td>0.0076</td>
</tr>
<tr>
<td>Berkshire</td>
<td>0.0137</td>
<td>0.0701</td>
<td>0.0113</td>
</tr>
<tr>
<td>Bob</td>
<td>0.0064</td>
<td>0.0467</td>
<td>0.0053</td>
</tr>
<tr>
<td>TEAM</td>
<td>0.0014</td>
<td>0.0217</td>
<td>0.0012</td>
</tr>
</tbody>
</table>
How much to invest?

- If we accept the objective to maximize long-run wealth, then the proportion of our wealth $p$ to put in an investment is

$$p = \frac{\mu - r_f}{\sigma^2}$$

$r_f$ is the risk-free rate of interest.

- Example suggests we’re more risk averse...
  - $\mu$ and $\sigma$ for the entire history of the market gives
  $$p = \frac{0.075}{0.040} = 1.75$$
  times wealth.

- Nonetheless, we ought to invest some fraction of our wealth in any asset for which we know $\mu \neq 0$ (short it if $\mu < 0$).
Which Investments?
Problem: So many choices?

- The simple analysis of how much to invest considers one asset, in isolation.

- Complications
  - Many many choices
  - Returns are correlated

- Role of dependence
  - Need to consider the correlation among the returns when investing in several
  - Messy problem of portfolio analysis is to anticipate correlations going forward.

- Theory from finance
  - Invest first in the market as a whole
  - Then consider other assets.
Dependence: Mutual Funds

- Regress growth in current year on prior growth
- Annual results for 1500 mutual funds
- “Statistically significant”

But the sign changes!

<table>
<thead>
<tr>
<th>Term</th>
<th>Estimate</th>
<th>Std Error</th>
<th>t Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td>18.634313</td>
<td>0.467789</td>
<td>39.83</td>
</tr>
<tr>
<td>Return 92</td>
<td>-0.49079</td>
<td>0.04243</td>
<td>-11.57</td>
</tr>
</tbody>
</table>

Explanation: 1,500 dependent observations...
Efficient Frontier

- Plot average return on SD of return for a collection of randomly formed portfolios

Leverage

Efficient Frontier

Mixing the “tangent” portfolio with cash obtains better performance

The tangent portfolio is the market portfolio.
Capital Asset Pricing Model

- Linear equation
  - Excess returns on an asset are related to those on whole market by a linear equation
  \[ R_t - r_f = \alpha + \beta (M_t - r_f) + \varepsilon_t \]
  - \( r_f \) is the risk-free rate
  - \( \beta = \text{Cov}(R_t - r_f, M_t - r_f)/\text{Var}(M_t - r_f) \)
  - \( \alpha = 0 \)

- Orthogonal
  - Divide risk into market and specific
  - Specific returns uncorrelated with market
    \[ (R_t - r_f) - \beta (M_t - r_f) = \alpha + \varepsilon_t \]

- If \( \alpha \neq 0 \)?
  - Intrinsic variation in asset has non-zero mean
  - Buy (or sell) some amount of it.
CAPM Regressions

All have substantial correlation with the returns on the whole market (market risk)

\[ M_{t} - r_{f} = \text{market excess return} \]
CAPM Residuals

Residuals often deviate from the usual assumptions of simple regression.
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CAPM Residuals

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Usual Test of Alpha

- Example: Exxon
- Regress out market risk, obtaining estimates of $\alpha$ and $\beta$.
  - $\beta = 0.56$
  - $\alpha = 0.0062$
- Test $H_0: \alpha = 0$
- Standard procedure relies on t-distribution to obtain p-value

<table>
<thead>
<tr>
<th></th>
<th>Estimate</th>
<th>SE</th>
<th>t</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alpha</td>
<td>0.0062</td>
<td>0.0023</td>
<td>2.68</td>
<td>0.0077</td>
</tr>
<tr>
<td>Beta</td>
<td>0.5623</td>
<td>0.0500</td>
<td>11.25</td>
<td>0</td>
</tr>
</tbody>
</table>
## Summary of Tests

<table>
<thead>
<tr>
<th></th>
<th>Estimate of Alpha</th>
<th>t</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exxon</td>
<td>0.0062</td>
<td>2.7</td>
<td>0.008</td>
</tr>
<tr>
<td>Berkshire</td>
<td>0.0103</td>
<td>3.1</td>
<td>0.002</td>
</tr>
<tr>
<td>Bob</td>
<td>0.0016</td>
<td>5.5</td>
<td>0.000</td>
</tr>
<tr>
<td>Team</td>
<td>-0.0008</td>
<td>-2.6</td>
<td>0.009</td>
</tr>
</tbody>
</table>

Do you believe these results?
Robust Testing
Testing Alpha

- Standard test procedure
  - Regress out the market
  - Test $H_0: \alpha = 0$ using t-statistic

- Model risk
  - Doubts about standard test.
  - What's the distribution of the t-statistic?
    Some investments produce returns that are far from Gaussian, with large outliers (fat tails)
  - Evident lack of independence in CAPM residuals
  - ARCH processes

- Nonetheless want a p-value
Martingale Test (CERT)

- Specific returns after removing market
  \[ w_t = (R_t - r_f) - \beta (M_t - r_f) = \alpha + \varepsilon_t \]

- Null hypothesis \( H_0: \alpha = 0 \)
  - Implies does not “beat the market”
  - Assume only that \( E(w_t|w_{t-1},w_{t-2},\ldots) = 0 \) (not nec. iid)

- Compound returns are non-negative martingale
  \[ C_t = (1+w_1)(1+w_2)\ldots(1+w_t) \quad t = 1,2,\ldots,n \]

- CERT p-value from Doob’s inequality
  \[ P(\text{max } C_1,\ldots,C_n \geq \gamma) \leq 1/\gamma \]

- Easy to use
  To reject \( H_0 \) at 0.05 level, compound returns have to exceed 20 during observed period
Example

“Residual” returns for Exxon,

$$(R_t - r_f) - b (M_t - r_f)$$

Since the martingale test does not depend on $n$, we can use finely spaced data that essentially reveal $\beta$ (if you believe it’s fixed!)
CERT Results

- Only Berkshire Hathaway rejects the null, and then we have to consider multiplicity.
Wrapping Up
Discussion

- **Multiplicity**
  A p-value of 1/20 does not overcome adjustments for multiplicity.

- **Bonferroni p-value**
  Multiply the p-value from martingale test by number of assets considered.
  
  - I bet that you have considered more than 4.

- **Power**
  The test is “tight” in the sense that there are processes you would not want to consider for which it gets the right answer, such as...
Bob Fund

How do you guarantee those 2% above benchmark returns?

Unobserved volatility

- \( r_t = \frac{1}{k} \) w.p. \( \frac{k}{k+1} \)
- \( r_t = -1 \) w.p. \( \frac{1}{k+1} \)
- \( \text{E}(r_t) = 0 \)

Example

- \( k = 19 \), so returns a bit more than 2% growth
- Smaller \( k \) give more exciting performance

For any choice of \( k \)

- \( P(C_t \text{ of Bob Fund} > 20) = \frac{1}{20} \)

Martingale test protects against the “until it happens” unobserved volatility
Summary

- Focus on returns, not cumulative value
- Remove market performance
  Regress out market from returns (CAPM)
- Watch for unseen volatility using robust test
  Martingale test (CERT)
- Adjust for multiplicity
  Bonferroni does fine, particularly since it’s hard to “count” the considered alternatives

Thanks!

www-stat.wharton.upenn.edu/~stine

Foster, Stine, Young (2008) “A martingale test for alpha”