Valuing Investments
A Statistical Perspective

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Overview

Principles
- Focus on returns, not cumulative value
- Remove market performance (CAPM)
- Watch for unseen volatility (peso problem)
- Adjust for multiplicity

How to evaluate...
- Investments as if they behave like familiar random processes.
- Plethora of choices offered by financial advisors
- Specific investments using data
Financial Data

- Examples
  - Indices
  - Portfolios
  - Mutual Funds
  - Hedge Funds
  - Commodities
  - Eurodollars
  - ...

- Time series without signal!

Case study in selection bias.
Multivariate co-integrated time series.

Chua, Foster, Ramaswamy, Stine (2007)
Mutual Funds

- Regress growth in current year on prior growth
- Annual results for 1500 mutual funds
- “Statistically significant”

But the sign changes!

Explanation: 1500 dependent observations...
Overall Market Performance


7% Annual growth

Data: Yahoo finance, Jan 1950 – Feb 2009, 710 months
Cumulative Returns?

Too easy to be deceived...

“Special”

Moral: Stick to returns...
Monthly Returns

Much simpler structure, almost iid...

\[ \frac{P_t - P_{t-1}}{P_{t-1}} \]

October 1987
“Black Monday”

August 1998
Long Term Capital

October 2008
Banking Crisis
Distribution of Returns

\[ \text{mean} = 0.0064, \quad s = 0.0415, \quad s^2 = 0.0017 \]

Fat tails more apparent in daily data.
The Dice Game
What makes a good investment?

Consider 3 investments...

<table>
<thead>
<tr>
<th>Investment</th>
<th>Average Annual Return</th>
<th>SD Annual Return</th>
</tr>
</thead>
<tbody>
<tr>
<td>Green</td>
<td>7.5%</td>
<td>20%</td>
</tr>
<tr>
<td>Red</td>
<td>71%</td>
<td>132%</td>
</tr>
<tr>
<td>White</td>
<td>0%</td>
<td>6%</td>
</tr>
</tbody>
</table>

Questions
- Which of these do you like, if any?
- How do you decide: risk versus return?
Hands-on Simulation

3 dice determine outcomes:

\[ W_t = (\text{Table Result}) \times W_{t-1} \]

<table>
<thead>
<tr>
<th>Outcome</th>
<th>Green</th>
<th>Red</th>
<th>White</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.8</td>
<td>0.06</td>
<td>0.9</td>
</tr>
<tr>
<td>2</td>
<td>0.9</td>
<td>0.2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1.05</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>1.1</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>1.2</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>1.4</td>
<td>3</td>
<td>1.1</td>
</tr>
</tbody>
</table>

Typical Results

- **Red** is “exciting” but generally loses value.
- **Green** offers steady growth.
- **White** goes nowhere.

*Green* is calibrated to match annual excess returns on US stock market.

*White* is calibrated to match returns on Treasury Bills.

We made up **Red**!
Occasional Results

- **Red** soars...
- In 20 rounds, the expected value of **Red** is
  \[1.71^{20} = 45,700\]
  times initial value
Something to ponder

Most simulations with the dice result in Red having lost most of its value.

A few simulations end with Red being fabulously wealthy, the “Warren Buffetts” of the class

In the long run, Red will lose (w.p. 1)

How can I recognize that Red will lose without waiting for it to happen?

Even so, how can I take advantage of Red?
A Special Opportunity!

While you are thinking about those dice, here’s a special opportunity...

The Bob Fund

Guarantees 2% excess annual returns above any benchmark you want. Guaranteed.

Rest assured, it’s not a Ponzi/Madoff scheme.

Contact me after the talk...
Investment Objective

- **Long-run wealth**
  \[ W_t = W_{t-1} (1+r_t) \]
  \[ = W_0 (1+r_1)(1+r_2) \ldots (1+r_t) \]

- **If the** \( r_t \) **are independent over time, then**
  \[ W_t \approx W_0 (1 + E(r_t) - \text{Var}(r_t)/2) \]

<table>
<thead>
<tr>
<th>Color</th>
<th>( E(r_t) )</th>
<th>( \text{Var}(r_t) )</th>
<th>( E(r_t) - \frac{\text{Var}(r_t)}{2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Green</td>
<td>0.075</td>
<td>((0.20)^2 = 0.04)</td>
<td>(.075 - .04/2 = .055)</td>
</tr>
<tr>
<td>Red</td>
<td>0.71</td>
<td>((1.32)^2 = 1.74)</td>
<td>-0.16</td>
</tr>
<tr>
<td>White</td>
<td>0</td>
<td>((0.06)^2 = 0.003)</td>
<td>-0.002</td>
</tr>
</tbody>
</table>

Volatility Drag

Can buy this one
Diversifying is good.

- Mix investments rather than leaving everything in one.
- **Pink** is a 50/50 mixture of Red & White.
  
  \[
  E(\text{Pink}) = E(0.5 \text{ Red} + 0.5 \text{ White}) \\
  = E(\text{Red})/2 = 0.355 \\
  \text{Var}(\text{Pink}) = \text{Var}(0.5 \text{ Red} + 0.5 \text{ White}) \\
  = \text{Var}(\text{Red})/4 = 0.435
  \]

- Long-run value of **Pink** is positive:
  
  \[
  E(\text{Pink}) - \text{Var}(\text{Pink})/2 = 0.14
  \]
  even though neither Red not White perform well taken separately.

Sacrifice **half** of the return to reduce the variance by 4.
Lessons from Dice Game

- Long-run value determined by
  \[ E(\text{return}) - (1/2) \text{Var(\text{return})} \]

- Over short horizons, a poor long-term investment might appear very attractive.

- Portfolios succeed by trading expected returns for reductions in variance
Cautions

- Real investments lack some properties of the investments in the dice simulation

- Independence
  - The dice fluctuate independently of one another.
  - The returns of Red are not affected by what happens to Green.

- Stability
  - The properties of the dice stay the same throughout the simulation. The chance for a good return on Red does not change.

- Parameters known
  - We know the properties of the random processes in the dice game.
Back to the Real World
Questions

Two fundamental questions

How much?

- How much of my wealth should I invest to meet my financial goals?

Which assets?

- Start with the whole-market index
- Which other investments in addition to index?
How much to invest?

- If we accept the objective to maximize long-run wealth, then the proportion of our wealth \( p \) to put in an investment is
  \[
  p = \frac{\mu - r_f}{\sigma^2}
  \]
  \( r_f \) is the risk-free rate of interest

- Example suggests we're more risk averse...
  - \( \mu \) and \( \sigma \) for the history of the market gives
    \[
    p = \frac{0.075}{0.040} = 1.75
    \]
    times wealth.

- Nonetheless, we ought to invest some fraction of our wealth in any asset for which we know \( \mu \neq 0 \) (short it if \( \mu < 0 \)).
Problem: So many choices?

- The simple analysis of how much to invest considers one asset, in isolation.
- Role of dependence
  - Need to consider the correlation among the returns when investing in several
  - Messy problem of portfolio analysis is to anticipate correlations going forward.
- Theory from finance
  - Invest first in the market as a whole
  - Then consider other assets.
Efficient Frontier

Plot average return on SD of return for a collection of randomly formed portfolios

Leverage

The tangent portfolio is the market portfolio.

Mixing the “tangent” portfolio with cash obtains better performance
Capital Asset Pricing Model

- Linear equation
  - Excess returns on an asset are related to those on whole market by a linear equation
    \[ r_t - r_f = \alpha + \beta (M_t - r_f) + \varepsilon_t \]
  - \( r_f \) is the risk-free rate
  - \( \beta = \text{Cov}(r_t-r_f, M_t-r_f)/\text{Var}(M_t-r_f) \)
  - \( \alpha = 0 \)

- Orthogonal
  - Intrinsic returns uncorrelated with market
    \[ (r_t - r_f) - \beta (M_t - r_f) = \alpha + \varepsilon_t \]
  - If \( \alpha \neq 0 \)?
    - Intrinsic variation in asset has non-zero mean
    - Buy (or sell) some amount of it.
Testing Alpha

Example: Berkshire-Hathaway

Regress out the market, obtaining estimates for $\alpha$ and $\beta$.

- $\beta = 0.722$
- $\alpha = 0.014$

Test $H_0: \alpha = 0$

Standard procedure relies on t-distribution to obtain p-value

| Term                     | Estimate  | Std Error  | t Ratio | Prob>|t| |
|--------------------------|-----------|------------|---------|-----|-----|
| Intercept                | 0.013962  | 0.003397   | 4.11    | <.0001* |
Testing Alpha

**Procedure**
- Regress out the market, obtaining estimates \( a \) for \( \alpha \) and \( b \) for \( \beta \).
- Test \( H_0: \alpha = 0 \) using regression estimates.

**Doubts?**
- What's the distribution of the t-statistic? Some investments produce returns that are far from Gaussian. Cannot rely on t-distribution.
- How to handle the issue of multiplicity? It is unlikely that we only consider only one other asset aside from the market as a whole. Methods (FDR, Bonferroni,...) require \( p \)-value.
Alternative Test for Alpha

- Returns after removing market
  \[ R_t = 1 + (r_t - r_f) - \beta (M_t - r_f) \]

- Null hypothesis \( H_0 \)
  The investment has \( \alpha = 0 \), so \( E(R_t) = 1 \). The alternative does not “beat the market”

- Compound these returns
  \[ C_t = R_1 R_2 \ldots R_t \geq 0, \quad t = 1,2,\ldots,n \]

- Test p-value
  \[ P(C_1,\ldots,C_n|H_0) \leq 1/\max(C_t) \]

- Easy to use
  To reject \( H_0 \) at 0.05 level, compound returns have to exceed 20 during observed period

Martingale Test

- Martingale
  Stochastic process \( \{X_t\} \) for which
  \[
  E(X_{t+1}|X_t,X_{t-1},X_{t-2}) = X_t
  \]

- Classic examples
  - Sum of coin tosses
  - Random walk
  - Martingale does not require independence

- Test for alpha treats compound returns \( C_t \) as a non-negative martingale with conditional expected value 1.

- Doob's martingale inequality
  \[
  P(\max(X_1,X_2,...,X_n) \geq \lambda) \leq E \frac{X_n}{\lambda}
  \]
Example

“Residual” returns for Berkshire-Hathaway,

\[(r_t - r_f) - b (M_t - r_f)\]

Note: since the martingale test does not care about \(n\), we can use finely spaced data that essentially reveal \(\beta\) (if you believe its fixed!)

Implied p-value

\[1/80 = 0.0125\]

not nearly so impressive as t-statistic
Discussion

Multiplicity
A p-value of 1/80 does not overcome even slight adjustments for multiplicity.

Bonferroni p-value
Multiply the p-value from martingale test by number of assets considered.
  I bet that you have considered more than 4.

Power
The test is “tight” in the sense that there are processes you would not want to consider for which it gets the right answer.
Bob Fund

- How do you guarantee those 2% above benchmark returns?

- Unobserved volatility
  - \( r_t = 1/k \) w.p. \( k/(k+1) \)
  - \( r_t = -1 \) w.p. \( 1/(k+1) \) 
    - busted
  - \( E(R_t) = 1 + E(r_t) = 1 \)

- Example
  - \( k = 19 \), so returns a bit more than 2% growth
  - Smaller \( k \) give more exciting performance

- For any choice of \( k \)
  - \( P(C_t \text{ of Bob Fund} > 20) = 1/20 \)

- Martingale test protects against the “until it happens” unobserved volatility
Summary

Principles

- Focus on returns, not cumulative value
- Remove market performance
  - Regress out market from returns
- Watch for unseen volatility
  - Martingale test
- Adjust for multiplicity
  - Bonferroni does fine, particularly since it’s so hard to “count” the considered alternatives

No free lunches or dinners!

Thanks!

www-stat.wharton.upenn.edu/~stine