Assignment Solutions (#10)

.9 (a) Let $X_i = \begin{cases} 1 & i^{th} \text{ urn is empty} \\ 0 & \text{otherwise} \end{cases}$, $i = 1, \dots, n$. Then

$$P(X_i = 1) = E X_i = \underbrace{\left(1 - \tfrac{1}{i}\right)}_{\uparrow \text{ miss on } i^{th} \text{ ball}} \left(1 - \tfrac{1}{i+1}\right) \cdots \left(1 - \tfrac{1}{n}\right) = \prod_{j=i}^{n} \left(1 - \tfrac{1}{j}\right)$$

$$E(\#\text{ of empty urns}) = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \left(\prod_{j=i}^{n} 1 - \tfrac{1}{j}\right)$$

(b) $$P(\text{no urn is empty}) = P(\text{ball } 1 \text{ in } 1) \cdot P(\text{ball } 2 \text{ in } 2) \cdots P(\text{ball } n \text{ in } n)$$
$$= 1 \cdot \tfrac{1}{2} \cdot \tfrac{1}{3} \cdots \tfrac{1}{n}$$

.12 (a) Let $X_i = \begin{cases} 1 & i^{th} \text{ person in line is a man standing next to woman} \\ 0 & \text{otherwise} \end{cases}$

$$P(X_i = 1) = \begin{cases} \underbrace{\tfrac{1}{2}}_{P(M)} \cdot \underbrace{\tfrac{n}{2n-1}}_{P(F \mid M)} & \text{at end of line} \\ \underbrace{\tfrac{1}{2}}_{P(M)} \cdot \underbrace{\left(1 - \tfrac{n-1}{2n-1} \cdot \tfrac{n-2}{2n-2}\right)}_{P(\quad \mid M)} & \text{not at end of line} \end{cases}$$

$$E[\#\text{ men with woman next}] = E\left[\sum_{i=1}^{2n} X_i\right]$$

$$= \underbrace{\cancel{2}\left(\tfrac{1}{\cancel{2}} \tfrac{n}{2n-1}\right)}_{\uparrow \text{ ends}} + (2n-2)\left(\tfrac{1}{2} \cdot \left(1 - \tfrac{n-2}{2n-1} \cdot \tfrac{1}{2}\right)\right)$$

$$= \frac{n}{2n-1} + (n-1)\frac{3n}{2(2n-1)} = \frac{n(3n-1)}{2(2n-1)}$$

(b) There are no ends, so

$$E[\#\text{ men with woman next}] = 2n\left(\tfrac{1}{2} \cdot \left(1 - \tfrac{n-2}{2(2n-1)}\right)\right)$$
$$= 3n^2 / 2(2n-1)$$

7.19 (a) Subtract 1 from geometric, or $\frac{1}{p_1} - 1$

(b) Let $X_i = \begin{cases} 1 & \text{catch type } i \text{ before type 1} \\ 0 & \text{otherwise} \end{cases}$. then

$$E[X_i] = P(X_i = 1) = P(\text{catch type } i \mid \text{catch } i \text{ or type } 1)$$
$$= \frac{p_i}{p_1 + p_i}$$

$$E[\#\text{ species caught before type } i] = \sum_{i=2}^{r} E X_i = \sum_{i=2}^{n} \frac{p_i}{p_1 + p_i}$$

7.34 (a) Let $X_i = \begin{cases} 1 & i^{th} \text{ husband next to wife} \\ 0 & \text{otherwise} \end{cases}$. then

$$E[\#\text{ husbands next to wife}] = E\left[\sum_{i=1}^{10} X_i\right]$$
$$= n\left(2/19\right)$$
$$= 10 \cdot 2/19$$

ie, she sits down, and he can sit in any of 19 spots, only 2 are next to her

7.36 Either use the binomial argument from class or a counting argument with indicators. the binomial is nicer.

$$(X+Y) \sim Bi(n, 1/3) \qquad X, Y \sim Bi(n, 1/6$$

$$\mathrm{Var}(X+Y) = \mathrm{Var}\,X + \mathrm{Var}\,Y + 2\,\mathrm{Cov}(X,Y)$$
$$= n\left(\tfrac{1}{6}\right)\left(\tfrac{5}{6}\right) + n\left(\tfrac{1}{6}\right)\left(\tfrac{5}{6}\right) + 2\,\mathrm{Cov}(X,Y)$$
$$n\left(\tfrac{1}{3}\right)\left(\tfrac{2}{3}\right) = 2n\tfrac{5}{36} + 2\,\mathrm{Cov}(X,Y)$$

$$\Rightarrow \mathrm{Cov}(X,Y) = -n/36$$

7.46 $\text{Cov}(I_1, I_2) = E(I_1 I_2) - (EI_1)(EI_2)$

$= P(\text{both win}) - P(1\text{ wins})\,P(2\text{ wins})$

condition on bank's roll

$= \sum_b P(\text{both win} \mid B=b)\,P(B=b) - \left(\sum_b P(1\text{ wins} \mid B=b)\,P(B=b)\right)^2$

$= \sum P(1\text{ wins} \mid B=b)\,P(2\text{ wins} \mid B=b)\,P(B=b) - \text{"}\downarrow\text{"}$

$= E\left[P(1\text{ wins} \mid B=b)^2\right] - \left(E\left[P(1\text{ wins} \mid B=b)\right]\right)^2$

$\geq 0 \quad \text{since} \quad EX^2 \geq (EX)^2$

7.48 (a) Geometric: X, Y $\qquad EX = \frac{1}{p} = \frac{1}{(1/6)} = 6$

(b) $E[X \mid Y=1] = E[\text{rolls to 5} \mid \text{first roll is 6}] = 6+1$ (i.e., just start over after first roll, memoryless)

(c) $E[X \mid Y=5] = 1 \cdot P(X=1 \mid Y=5) + 2 \cdot P(X=2 \mid Y=5) + \cdots$

$= 1 \cdot \frac{1}{5} + 2 \cdot \left(\frac{4}{5}\right)\left(\frac{1}{5}\right) + 3\left(\frac{4}{5}\right)^2\left(\frac{1}{5}\right) + 4\left(\frac{4}{5}\right)^3\left(\frac{1}{5}\right) + 0 + \left(\frac{4}{5}\right)^4(5+6)$

did not roll a 6 here since $Y=5$

restart geometric

7.53 $E(D) = E[\text{Days}] = E[\text{Days} \mid \text{Door 1}]\,P[\text{Door 1}] + E[\text{Days} \mid \text{Door 2}]\,P(\text{Door 2}) + E[\text{Days} \mid \text{Door 3}]\,P(\text{Door 3})$

$P[\text{Door 1}] = .5$, $P(\text{Door 2}) = .3$, $P(\text{Door 3}) = .2$

$= (2+ED)(.5) + (4+ED)(.3) + 1(.2)$

$ED = 1 + ED/2 + 1.2 + (ED)(.3) + (.2)$

$.2\,ED = 1 + 1.2 + .2$

$\Longrightarrow ED = \frac{2.4}{.2} = 12$

1.55 Assume flock has $N=n$ ducks. Given $N=n$, the expected number hit is then

$$E[\underbrace{H}_{\#\text{hit}} \mid N=n] = E\sum_{d=1}^{n} X_d \qquad X_i = \begin{cases} 1 & \text{Duck } d \text{ is hit} \\ 0 & \text{missed} \end{cases}$$

$$P(X_d = 1) = E[X_d] \text{ and } \Rightarrow P(X_d=1) = 1 - P(X_d=0)$$
$$= 1 - P(\text{all 10 miss duck } d)$$
$$= 1 - \left(\tfrac{.6}{n}\right)^{10}$$

$\Uparrow$ $P(\text{pick}) \cdot P(\text{hit}|\text{pick})$
$\frac{1}{n} \cdot .6$

Define $H=0$ if no ducks appear.

$$\therefore \quad EH = \sum_{n=0}^{\infty} E[H \mid N=n]\, P[N=n]$$
$$= 0 + \sum_{n=1}^{\infty} n\left(1-\left(\tfrac{.6}{n}\right)^{10}\right)\left(\frac{e^{-6}\, 6^n}{n!}\right)$$

$\Uparrow$ Poisson prob of n in flock

1.68 (a) $P(0 \text{ accidents}) = P(X=0 \mid \lambda=2)P(\lambda=2) + P(X=0 \mid \lambda=3)P(\lambda=3)$
$= e^{-2}(.6) + e^{-3}(.4) = p_0$

(b) $P(3 \text{ accidents}) = P(X=3 \mid \lambda=2)P(\lambda=2) + P(X=3 \mid \lambda=3)P(\lambda=3)$
$= \frac{e^{-2}(2)^3}{3!}(.6) + \frac{e^{-3}(3)^3}{3!}(.4) = p_3$

(c) $$P(3 \mid 0) = \frac{P(0 \text{ in year 1 and } 3 \text{ in year 2})}{P(0 \text{ in year 1})}$$

you learn about λ since none in first year

$$= \frac{e^{-2}\cdot e^{-2}(2)^3/3!\,(.6) + e^{-3}\cdot e^{-3}(3)^3/3!\,(.4)}{p_0}$$

first year   second year

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