Investing in More Complex Portfolios

Administrative Things

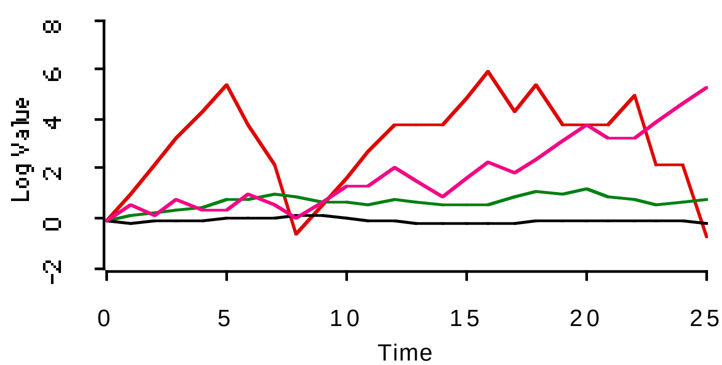
- Dice investments
  - Table below shows mean and variability for all 4 "investments".
  - Remember that for figuring out what happens with "Pink" that

    $Var(aX + bY) = a^2 \, Var\, X + b^2 \, Var\, Y$ for uncorrelated r.v.s.

  Averaging has a big effect on the variance.

Color Die	Annual Return	Variability (SD)
Green	7.5%	20%
Red	71%	130%
White	0%	5%
Pink	35.5%	$\sqrt{(.0025 + 1.69)/4} = 0.650$

- Typical plot of log of the value of the different investments. "Red" is volatile.



  - "Your mileage may vary."

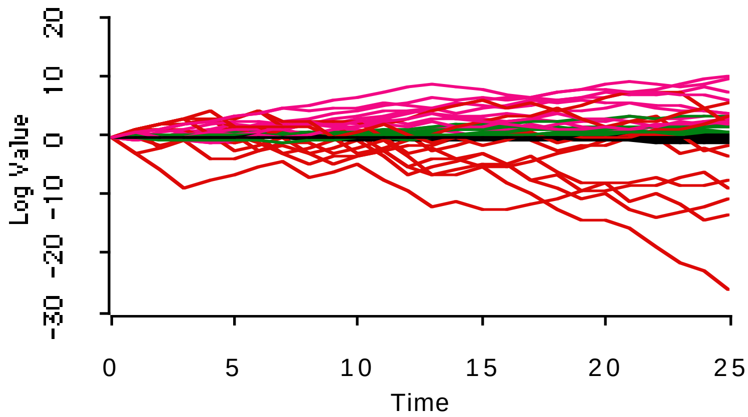
- How should we anticipate what these three investments will do?
  - Plot of simulated collection of portfolios (at the top of the next page)

    - Shows what happened for 10 simulated series

    - Red is quite volatile.

    - White and green are almost invisible at center, with pink positive.

- Keep in mind that these are plots of the “raw” investments, not necessarily your investment if you are leveraged.



- Comparison in terms of our CEV calculations… Avg/Var

Color Die	Average	Variance	Avg/Var
Green	0.075	0.04	1.875
Red	0.710	1.69	0.420
White	0.000	0.0025	0
Pink	0.355	0.433	0.820

- Of the original three dice investments

CEV values suggest that you buy a little “Red”, but a lot of “Green”.

Since “Green” is leveraged by about 2/1 whereas “Pink” is not, the plot above shows the performance of an almost optimal amount of “Pink”, but not of the optimal Green portfolio (which would be have to get scaled by about 2).

- Curiously, “Pink”, the 50/50 mixture of “Red” and “White”, two that are not so appealing when bought separately, is very appealing once combined.

Why does the mixture of two unfavorable investments seem favorable?

What is it about these two that makes the “average” investment (pink) look so attractive?

➢ What were the values of your simulated returns?

- We’ll enter some of these in class and look at the mean and variance of the results.

- Theoretical analysis is most easy, and appropriate, with the data on a log scale. Taking logs turns the product of the returns into a sum.

$$\text{Log } W_t = \text{Log}(\ W_0(1+R_1 * 1+R_2 * \ldots * 1+R_t)\ )$$
$$= \text{Log } W_0 + \text{Log } 1+R_1 + \text{Log } 1+R_2 + \ \ldots + \text{Log } 1+R_t$$

so that

$$\text{E Log } W_t = \text{Log } W_0 + t \text{ E Log } 1+R_1$$

and similarly for the variance (at least in this "simulation" world we can be sure that the variance is indeed constant)

$$\text{Var Log } W_t = t \text{ Var Log } 1+R_1$$

- Here's a summary of what happens on average

  - These are calculated directly (only six possible outcomes).

  - At other times you will see these calculations based on the simple approximation

$$\log(1+x) \approx x - x^2/2 \qquad \text{for} \qquad x \approx 0.$$

This approximation does suggest how the variance enters the calculation of the mean of the logs, and is *very similar* to our CEV expressions.

Color Die	E 1+R	Var 1+R	E Log	Var Log
Green	1.075	0.04	0.055	0.030
Red	1.710	1.69	-0.19	2.300
White	1.000	0.0025	-0.001	0.003
Pink	1.355	0.433	0.14	≈0.400

- Multiply up by t=25 for 25 rounds of the game (log scale)

Color Die	E Log	Var Log
Green	1.375	0.75
Red	-4.75	58
White	-0.025	0.075
Pink	2.75	10

- How do these compare to your results?

When we add Log(W_0) to 2.75 for Pink, we get 10.4. This compares to the average log of 10.3 in your results.

- Note that we know of no real investment with Red's characteristics. Green is designed to parallel the real returns on the market, and White is the net return on the so-called risk-free investment (net of inflation).

Today's Topics

- ➢ Allocating your money within the market
  - As a combination of the value-weighted index and some decile.
    - Just want to finish/review this topic from last time.
  - As a combination of the value-weighted index and a mixture of deciles.
    - Allow you to buy a mixture of deciles (like "pink") rather than just one.
- ➢ Technical difficulty
  - How to find the variance of a portfolio of correlated investments.
- ➢ Statistical methods for constructing uncorrelated investments
  - Regression for making two uncorrelated investments.
  - *Principal components* for more than two.

Review from Last Time

- ➢ How much to invest in an instrument?
  - Suppose there is only one risky investment out there (we did this using the value-weighted index). You can put your money into this instrument, or "leave it in the bank" earning the risk-free rate.
  - If k = *wealth* in our utility, then we maximize the utility of our future wealth by investing p times our wealth in this instrument, where p is determined by

$$p = E(\text{return}) / \text{Var}(\text{return}) .$$

- If this instrument is the value-weighted stock market, from our analysis of the historical data since 1926, this gives $p = 2$.

➢ Assumptions

- Only one speculative instrument around.
- Constant variance/volatility (we know that this is not the real case).
- Borrowing at the risk-free rate (when you can't, the optimal investment in the market was in fact closer to what a more risk-averse investor will choose to do).

➢ Implication for our value of k – we are more risk averse than k = *wealth*.

- "Most of us" do not put double our wealth into the market.
- Thus, for many of us, the value of k is probably smaller, indicating more risk aversion.

Allocating Your Market Investment

➢ Next question

- Once we decide to put money into the market, how should we divide the money we invest in the market among the stocks that are available?

➢ Answer (that I hope to demonstrate)

- Just buy the value-weighted market index.

➢ Simplified market with VW index and 10 decile "investments"

- The first decile are the smallest 10% of the companies, and the 10^{th} decile has the largest companies. The10th decile captures about 3/4 of the total market return.
- The return for each decile is the return on a value-weighted portfolio for that decile.

➢ A yet more simplified choice

- You can pick the market *plus any one* of the deciles.

- To maximize our utility, again with *k=wealth*, would we ever purchase any of the decile instrument? No!

  Aside – What would happen with a smaller value for *k*?

- Logarithmic utility leads us to maximize the criterion (as before)
$$\max_p E\, W_{t+1} - \text{Var}\, W_{t+1}/(2W_t)\ ,$$
where now the wealth is based on the return of our portfolio.

➢ Simple two-item portfolio of self-financing investments

- Consider a portfolio that invests a multiplier p_v of our current wealth in the value-weighted index and p_d in one of these 10 deciles.

- Each item is self-financing in the sense that we borrow at the risk-free rate to purchase each investment, and net the excess return above risk-free over the month. Thus p_v and p_d need not be proportions (add to one) of our wealth and are simply constants.

- Ignoring issues of inflation, the wealth from these investments evolves as
$$W_{t+1} = W_t\ (\ 1 + p_v\ vw_t + p_d\ d_t)$$
where the lower case symbols denote the excess returns
$$vw_t = VW_t - RF_t \qquad\qquad d_t = D_t - RF_t$$

➢ Utility of the two-item portfolio requires means and variances.

- Expected wealth is easy to find
$$E\ W_{t+1} = W_t\ (\ 1 + p_v\ E(vw_t) + p_d\ E(d_t))$$

- But the variance of our wealth depends upon the covariance between the two data series.
$$\text{Var}\ W_{t+1} = W_t^2\ (\ p_v^2\ E(vw_t) + p_d^2 E(d_t) + 2p_v p_d \text{Cov}(vw_t, d_t)$$

- Implication of this covariance is that we cannot look at the two investments separately, but have to consider their joint behavior.

Building Uncorrelated Instruments

➢ Uncorrelated investments are simpler to understand.

- If X_1 and X_2 are excess returns for two instruments that are not correlated, then we could determine the wealth to invest in each separately, and the optimal solution is to choose the multiplier as before

$$p_i = E\ X_i \ /\ Var(X_i).$$

➢ Use regression

- Build a new uncorrelated instrument to go with the value-weighted index,

$$d_t - b\ vw_t$$

where the constant b is chosen from the regression of d_t on vw_t. This new series measures what the decile offers that is not already part of the vw index.

Don't subtract off the constant (and get the residuals) since the new investment would have to have mean zero.

- Recall that a regression slope is just Cov(X,Y)/Var(X), so that the fitted slope is the so-called *beta coefficient* for the decile return.

- The covariance is zero because that's what regression does: the residuals from each simple regression are uncorrelated with the predictor. They are correlated with each other.

Variable	ExcessVW	Resid Excess1	Resid Excess2
ExcessVW	1.00	-0.00	-0.00
Resid Excess1	-0.00	1.00	0.91
Resid Excess2	-0.00	0.91	1.00
Resid Excess3	-0.00	0.85	0.91
Resid Excess4	-0.00	0.81	0.88
Resid Excess5	-0.00	0.76	0.84
Resid Excess6	-0.00	0.72	0.80
Resid Excess7	-0.00	0.61	0.68
Resid Excess8	-0.00	0.47	0.55
Resid Excess9	-0.00	0.39	0.44
Resid Excess10	-0.00	-0.67	-0.74

➢ How much do we want to buy of the excess value-weighted returns and any one of these decile based instruments that are uncorrelated with the vw returns?

- We only want to buy instruments with positive return. We'll need Bonferroni here since we will be checking 11 things (vw and the 10 deciles).

The critical p-value is $0.05/11 \approx .0045$.

- For the excess value-weighted item, we have reproduced our previous results. The mean return is significantly positive ($p = .0003$) and the ratio of mean to variance is $.0068/.0030 = 2.26$.

- For each of these decile instruments, only the residual series from the first decile has a 95% confidence interval for the mean return that does not include zero.

Mean	0.0053
Std Dev	0.0748
Std Error Mean	0.0025
Upper 95% Mean	0.0103
Lower 95% Mean	0.0003

	t Test
Test Statistic	2.0938
Prob > |t|	0.0366

- Its p-value (p=0.037) is larger, though, than our Bonferroni value, so we cannot conclude it has significantly positive return either.

➢ Conclusion

- When offered a pair (VW and an uncorrelated decile based investment), we don't want to buy any of these new instruments. We'll just buy the VW index as before.

Working with a More Complex Market

➢ A richer class of investment options

- You can buy the value-weighted index and any of a collection of weighted combinations of the deciles.

- Remember "pink"

  Just because you did not want to buy any one of these uncorrelated deciles is no reason to believe that you might not want to purchase a mixture of these deciles.

  Note: averaging will not be quite so beneficial here since the series are correlated. "White" and "Red" were independent.

➢ Problem

- We had two uncorrelated investments so that again the analysis was simplified and hopefully more plausible.

- How can we construct combinations of the deciles that are uncorrelated with the value-weighted index and uncorrelated with each other as well?

- Start from the residual series constructed by regressing out the vw series (but leaving in the constant term).

$$(d_{1t} - b_1\ vw_t),\ (d_{2t} - b_2\ vw_t), \ldots,\ (d_{10t} - b_{10}\ vw_t)$$

- We know that these are highly correlated, however, so that the calculation of the variance of our wealth needed to figure out the right proportions depends upon covariances and such. What we need are a set of uncorrelated investments so that, once again, we simply look at the mean/var for each.

➢ Solution: use a method called principal components to construct the set of uncorrelated investments.

- Principal components is a procedure that when given a collection of observed random variables produces a set of new random variables which are uncorrelated.

- The new random variables are weighted sums of the inputs, just what we need for our analysis.

➢ JMP makes this task pretty easy.

- Start from the correlation tool, with all 10 residual decile series selected.

- Choose the principal components option, using covariances, from the check box at the bottom of the correlation window.

- Portion of the output is shown below

Principal Components

On Covariance Matrix

EigenValue:	0.0127	0.0011	0.0003
Percent:	86.2820	7.2449	2.3248
CumPercent:	86.2820	93.5269	95.8516
Eigenvectors:			
Resid Excess1	0.636	-0.618	0.435
Resid Excess2	0.471	-0.037	-0.489
Resid Excess3	0.382	0.185	-0.482
Resid Excess4	0.304	0.315	-0.081
Resid Excess5	0.240	0.324	0.140
Resid Excess6	0.205	0.348	0.146
Resid Excess7	0.148	0.336	0.318
Resid Excess8	0.099	0.325	0.326
Resid Excess9	0.059	0.185	0.281
Resid Excess10	-0.039	-0.068	-0.068

The "eigenvalues" are the variances of the new variables, and the "eigenvectors" shown below are the weights used to build them. Thus, the first new variable has variance 0.0127 and is formed as

$$0.636(d_{1t} - b_1\, vw_t) + 0.471(d_{2t} - b_2\, vw_t) + \ldots - 0.039\, (d_{10t} - b_{10}\, vw_t)$$

- Messy JMP thing

  JMP subtracts off the mean from the components, so we have to edit the formula or the mean of the new variables is exactly zero. Having a zero mean is "helpful" for some problems, but not what we want here.

➢ Do we want to purchase any of these?

- Again, we first check to see if any have significantly positive returns.
- Using the Bonferroni rule again (now what's the right divisor: 10 or 20???) with the relatively soft rule of $p < .005$ finds only two (Prin2 and Prin3) with 95% intervals that do not contain the mean, and for these we get

	t Test
Test Statistic	-2.5350
Prob > |t|	0.0114

	t Test
Test Statistic	2.6447
Prob > |t|	0.0083

- Close, but not quite significant.

  We have to look at Prin2 even though its mean is negative because we could simply "short" this investment and potentially come out ahead.

➢ OK, but how do I know there is not a "pink" mixture somewhere out there among these 10 uncorrelated investments? After all, "White" and "Red" were also uncorrelated.

- How do I know that a mixture these will not somehow be better?
- Is it enough to show that I do not what them individually?

Next Time

➢ Hedging and avoiding risk.