

## Homework #3

## Chapter 3, Shumway and Stoffer

## 3.2 AR(1) process with starting values, initialization

- (a) Write the process as

$$x_t = w_t + \phi w_{t-1} + \phi^2 w_{t-2} + \cdots + \phi^t w_1$$

The mean is the sum of the means,  $\mathbb{E} x_t = 0$  and the variance is

$$\text{Var} x_t = \sigma^2 \sum_{j=0}^t \phi^{2j} = \sigma^2 \frac{1 - \phi^{2(t+1)}}{1 - \phi^2}.$$

Hence, because the variance depends on  $t$ ,  $x_t$  is not stationary.

- (b) Part (a) gives the variance. The covariance is the collection of overlapping terms,

$$\begin{aligned} \text{Cov}(x_t, x_{t-h}) &= \text{Cov} \left( \sum_{j=0}^{t-h} \phi^{j+h} w_{t-h-j}, \sum_{j=0}^{t-h} \phi^j w_{t-h-j} \right) \\ &= \phi^h \text{Var } x_{t-h} \end{aligned}$$

The correlation is thus

$$\text{Corr}(x_t, x_{t-h}) = \frac{\text{Cov}(x_t, x_{t-h})}{(\text{Var } x_t \text{Var } x_{t-h})^{1/2}} = \phi^h \left( \frac{\text{Var } x_{t-h}}{\text{Var } x_t} \right)^{1/2}$$

- (c) Take the limit in the expression for the variance in part (a). In the limit, the role of the starting value vanishes since  $\phi^h \rightarrow 0$ .
- (d) Use normals as the  $w_t$  and discard some initial “burn in” period of data. In practice, when this procedure is used, the length of the burn-in is determined by the memory of the process which in turn is controlled by the proximity of the zeros of  $\phi(z)$  to the unit circle. Modern techniques avoid this by forming an initial block of  $p$  observations which have the stationary distribution – albeit only in the Gaussian case.
- (e) Start the process in its stationary form. This process is now stationary without the need for an approximate burn-in period. It’s easy to do in the AR(1) case, but requires a matrix factorization (Cholesky decomposition of  $\Gamma_p$  for  $p > 1$ ). For this problem, write

$$x_t = \sum_{j=0}^{t-2} \phi^j w_t + \phi^{t-1} x_1.$$

This has constant mean zero, and its variance does not depend on  $t$ :

$$\text{Var}(x_t) = \sigma_w^2 \frac{1 - \phi^{2(t-1)}}{1 - \phi^2} + \sigma_w^2 \frac{\phi^{2(t-1)}}{1 - \phi^2} = \sigma_w^2 \frac{1}{1 - \phi^2}$$

The covariances are also invariance of  $t$  (for  $|h| < t$ ),

$$\text{Cov}(x_t, x_{t-h}) = \text{Cov}\left(\sum_{j=0}^{h-1} \phi^j w_{t-j} + \phi^h x_{t-h}, x_{t-h}\right) = \text{Var}(x_{t-h})$$

which was shown to be stationary previously.

### 3.3 Polynomial zeros

Use R to find the zeros of the polynomial (*i.e.*, use the function `polyroot`).

(a)  $x_t = 0.8x_{t-1} - 0.15x_{t-2} + w_t - 0.3w_{t-1}$

$$\phi(z) = 1 - 0.8z + 0.15z^2 \Rightarrow z_1 = 2, z_2 = 1/0.3$$

$$\theta(z) = 1 - 0.3z \quad z_1 = 1/0.3$$

The process is causal and invertible as the zeros lie outside the unit circle. The zero of the moving average cancels one of those for the AR, reducing to an AR(1) process.

(b)  $x_t = x_{t-1} - 0.5x_{t-2} + w_t - w_{t-1}$

$$\phi(z) = 1 - z + 0.5z^2 \Rightarrow z_1 = 1 - i, z_2 = 1 + i$$

$$\theta(z) = 1 - z \quad z_1 = 1$$

The process is causal, but not invertible.

### 3.7 ARMA(1,1)

The point of this exercise is to see the qualitative differences among the autocorrelations and partial autocorrelations of the AR(1), MA(1) and ARMA(1,1) processes (*i.e.*, to appreciate what's in Table 1 on page 109).

In particular, the covariances of the ARMA(1,1) process are very similar to those of an AR(1), decaying geometrically past the initial values:  $\gamma(1)$ ,  $\gamma(k) = \phi\gamma(k-1)$  for  $k = 2, 3, \dots$ . The Yule-Walker equations provide the starting values for this recursion. Multiply both sides of the equation

$$x_t = \phi_1 x_{t-1} + \theta_1 w_{t-1} + w_t$$

by  $x_t, x_{t-1}, \dots$  and take expectations (use the infinite moving average form  $x_t = \sum \psi_j w_{t-j}$  as well):

$$\gamma(0) = \phi_1 \gamma(1) + (1 + \theta_1 \psi_1) \sigma^2$$

$$\gamma(1) = \phi_1 \gamma(0) + \theta_1 \sigma^2$$

$$\gamma(2) = \phi_1 \gamma(1)$$

$$\gamma(h) = \phi_1 \gamma(h-1), h = 3, 4, \dots$$

### 3.8 Simulate ARMA(1,1)

The point of this exercise is to notice that the estimated correlations don't always look very much like the actual autocorrelations. Hence, using Table 1 is a lot harder than you might think unless the series is hundreds of points long.

### 3.9 Cardio mortality, data analysis

- (a) I was curious whether any of you would extend the correlation function far enough to see the clear seasonal oscillation around 52 weeks. The data have a strong annual cycle.
- (b) The predictions revert to the mean quickly, with rather wide intervals.
- (c) The forecast errors are clearly dependent. In particular, even if we knew the model order, its parameters, and had infinite amounts of data, the errors are

$$\begin{aligned}
 y_{n+1} - \hat{y}_{n+1} &= w_{n+1} \\
 y_{n+2} - \hat{y}_{n+2} &= \psi_1 w_{n+1} + w_{n+2} \\
 y_{n+3} - \hat{y}_{n+3} &= \psi_2 w_{n+1} + \psi_1 w_{n+2} + w_{n+3}
 \end{aligned}$$

and so forth. The presence of estimated coefficients adds further dependence, but the order of that effect is dominated by the role of the  $w$ s.

- (d) Bonferroni adjustment is a start (*i.e.*, set the coverage to  $1-\alpha/4$ , say) and works even though the implied tests are dependent. You still get 95% coverage even though the intervals are correlated. Make sure that you aren't using independence of the coverage of the intervals; as noted in "c", they are dependent.

### 3.31 Global temperature deviations, data analysis

The standard diagnostics show that the series ought to be differenced. I let arima fit a model with differencing to build the predictions since this left the messy task of integrating up the predictions to the R software. There are problems, however, with R's calculations for an integrated model (check out the book's web site).

## Forward-Backward Question

- (a) The first task is to write the residual (estimate of the error) at step  $p+1$  as a function of the residual at step  $p$ . This requires substituting the expression from the Levinson recursion into the calculations

$$\phi_{p+1,j} = \phi_{p,j} - \phi_{p+1,p+1}\phi_{p,p-j}, \quad j = 1, \dots, p. \quad (1)$$

As vectors, define  $\phi_{p+1,-1} = (\phi_{p+1,1}, \dots, \phi_{p+1,p})'$  (without the last one) and write

$$\phi_{p+1,-1} = \phi_p - \phi_{p+1,p+1}\tilde{\phi}_p, \quad (2)$$

where the tilde indicates a vector in reverse order. The error at time  $t$  and step  $p+1$  is then

$$\begin{aligned}
 w_{p+1,t} &= x_t - \phi_{p+1,1}x_{t-1} - \dots - \phi_{p+1,p}x_{t-p} - \phi_{p+1,p+1}x_{t-p-1} \\
 &= x_t - (x_{t-1}, \dots, x_{t-p})(\phi_p - \phi_{p+1,p+1}\tilde{\phi}_p) - \phi_{p+1,p+1}x_{t-p-1} \\
 &= x_t - x_{t-1}^{t-p'}\phi_p - \phi_{p+1,p+1}(x_{t-p-1} - x_{t-p}^{t-1'}\phi_p) \\
 &= w_{p,t} - \phi_{p+1,p+1}\tilde{w}_{p,t-p-1}
 \end{aligned}$$

- (b) To interpret the expression in (a), consider adding the variable  $x_{t-p-1}$  to the  $\text{AR}(p)$  model. The partial regression of  $x_t$  on  $x_{t-1}, \dots, x_{t-p}$  leaves residuals  $w_{p,t}$ . The partial regression of the added variable  $x_{t-p-1}$  on  $x_{t-p}, x_{t-p+1}, \dots, x_{t-1}$  uses the coefficients in reverse order and produces residuals  $\tilde{w}_{p,t-p-1}$ . The partial regression of  $w_{p,t}$  on  $\tilde{w}_{p,t-p-1}$  has coefficient  $\phi_{p+1,p+1}$  and in turn produces the residuals found in (a).