Examples of Stationary Time Series

Overview

1. Stationarity
2. Linear processes
3. Cyclic models
4. Nonlinear models

Stationarity

Strict stationarity (Defn 1.6) Probability distribution of the stochastic process $\{X_t\}$is invariant under a shift in time,

$$\begin{aligned}\mathbb{P}(X_{t_1} \le x_1, X_{t_2} \le x_2, \ldots, X_{t_k} \le x_k) &= F(x_{t_1}, x_{t_2}, \ldots, x_{t_k}) \\ &= F(x_{h+t_1}, x_{h+t_2}, \ldots, x_{h+t_k}) \\ &= \mathbb{P}(X_{h+t_1} \le x_1, X_{h+t_2} \le x_2, \ldots, X_{h+t_k} \le x_k)\end{aligned}$$

for any time shift h and x_j.

Weak stationarity (Defn 1.7) (aka, second-order stationarity) The mean and autocovariance of the stochastic process are finite and invariant under a shift in time,

$$\mathbb{E}\, X_t = \mu_t = \mu \qquad \mathrm{Cov}(X_t, X_s) = \mathbb{E}\,(X_t - \mu_t)(X_s - \mu_s) = \gamma(t, s) = \gamma(t-s)$$

The separation rather than location in time matters.

Equivalence If the process is Gaussian with finite second moments, then weak stationarity is equivalent to strong stationarity. Strict stationarity implies weak stationarity only if the necessary moments exist.

Relevance Stationarity matters because it provides a framework in which averaging makes sense. Unless properties like the mean and covariance are either fixed or "evolve" in a known manner, we cannot average the observed data.

What operations produce a stationary process? Can we recognize/identify these in data?

Moving Average

White noise Sequence of uncorrelated random variables with finite variance,

$$\mathbb{E}\, W_t = \mu \overset{\text{often}}{=} 0 \qquad \mathrm{Cov}(W_t, W_s) = \begin{cases} \sigma_w^2 \overset{\text{often}}{=} 1 & \text{if } t = s, \\ 0 & \text{otherwise} \end{cases}$$

The input component ($\{X_t\}$ in what follows) is often modeled as white noise. Strict white noise replaces uncorrelated by independent.

Moving average A stochastic process formed by taking a weighted average of another time series, often formed from white noise. If we define $\{Y_t\}$ from $\{X_t\}$ as

$$Y_t = \sum_{i=-\infty}^{\infty} c_i\, X_{t-i}$$

then $\{Y_t\}$ is a moving average of $\{X_t\}$. In order to guarantee finite mean, we require $\{c_i\} \in \ell_1$, the space of absolutely summable sequences, $\sum |c_i| < \infty$. In order for $\{Y_t\}$ to have second moments (if the input series $\{X_t\}$ has second moments), then $\{c_i\} \in \ell_2$. (ℓ_2 is the space of all square summable sequences, those for which $\sum c_i^2 < \infty$.)

Examples

trival	$Y_t = X_t$
differences	$Y_t = X_t - X_{t-1}$
3-term	$Y_t = (X_{t+1} + X_t + X_{t-1})\,/3$
one-sided	$Y_t = \sum_{i=0}^{\infty} c_i X_{t-i}$

Examples in R The important commands to know are `rnorm` for simulating Gaussian white noise and `filter` to form the filtering. (Use the concatenation function `c` to glue values into a sequence.

Note the practical issue of *end values*: What value do you get for y_1 when forming a moving average? Several approaches are

1. Leave them missing.
2. Extend the input series by, say, back-casting $x_0, x_1, \ldots$ (such as by the mean or fitting a line).
3. Wrapping the coefficient weights (convolution).

4. Reflecting the coefficients at the ends.

Question: For what choices of the weights c_j does the moving average look "smoother" that the input realization?

Stationarity of the mean of a moving average $\{Y_t\}$ is immediate if $E\,X_t = \mu_x$ and the sum of the weights is finite. For $\{Y_t\}$ to be second-order stationary, then (assume that the mean of $\{X_t\}$ is zero, implying that the mean of $\{Y_t\}$ is also zero and that the weights are square summable)

$$\gamma_y(t,t{+}h) = \mathbb{E}\,Y_t\,Y_{t+h} = \mathbb{E}\sum_{i,j} c_i c_j X_{t-i} X_{t+h-j} = \sum_{i,j} c_i c_j \gamma_x(t{-}i, t{+}h{-}j)$$

If the input is weakly stationary, then

$$\gamma_y(h) = \sum_{i,j} c_i c_j \gamma_x(t-i, t+h-j) = \sum_{i,j} c_i c_j \gamma_x(h-j+i)$$

If it also happens that the input is white noise, then the covariance further simplifies to a "lagged inner product" of the weights used to construct the moving average,

$$\gamma_y(h) = \sigma_x^2 \sum_i c_i c_{i+h} \qquad (1)$$

Remark The expression (1) can be thought of as a factorization of the covariance matrix associated with the stochastic process $\{Y_t\}$. First, imagine writing $\{Y_t\}$ as a matrix product

$$Y = C\,X$$

where the infinite length vectors X and Y stack the elements of the two prcesses

$$X' = (\ldots, X_{-2}, X_{-1}, X_0, X_1, X_2, \ldots) \text{ and } Y' = (\ldots, Y_{-2}, Y_{-1}, Y_0, Y_1, Y_2, \ldots)$$

and C is an infinite-dimensional, square, diagonal matrix with c_0 along the diagonal arranged by stacking and shifting the coefficients as

$$C = \begin{pmatrix} \vdots \\ \ldots, c_1, c_0, c_{-1}, c_{-2}, c_{-3} \ldots \\ \ldots, c_2, c_1, c_0, c_{-1}, c_{-2}, \ldots \\ \ldots, c_3, c_2, c_1, c_0, c_{-1}, \ldots \\ \vdots \end{pmatrix}$$

If the input is white noise, then (1) represents the covariance matrix Γ as the product

$$\Gamma_y = \sigma_x^2 C C'$$

with Γ holding the covariances

$$\Gamma_{y,ij} = \gamma_y(i-j)$$

Recursive Processes (Autoregression)

Feedback Allow past values of the process to influence current values:

$$Y_t = \alpha Y_{t-1} + X_t$$

Usually, the input series in these models would be white noise.

Stationarity To see when/if such a process is stationary, use back-substitution to write such a series as a moving average:

$$\begin{aligned} Y_t &= \alpha(\alpha Y_{t-2} + X_{t-1} + X_t \\ &= \alpha^2(\alpha Y_{t-3} + X_{t-2}) + X_t + \alpha X_{t-1} \\ &= X_t + \alpha X_{t-1} + \alpha^2 X_{t-2} + \cdots \end{aligned}$$

Stationarity requires that $|\alpha| < 1$. If $\alpha = 1$, then you have random walk if $\{X_t\}$ consists of independent inputs.

Linear The ability to represent the autoregression (in this case, a first-order autoregression) as a moving average implies that the autoregression is a linear process (albeit, one with an infinite sequence of weights).

Cyclic Processes

Random phase model Define a stochastic process as follows. Let U denote a random variable that is uniformly distributed on $[-\pi, \pi]$, and define (Here R is a constant, but we could allow it to be an independent r.v. with mean zero and positive variance.)

$$X_t = R\cos(\phi\, t + U)$$

Each realization of $\{X_t\}$ is a single sinusoidal "wave" with frequency ϕ and amplitude R.

Trig sums are always easier in complex notation unless you use them a lot. You just need to recall Euler's form (with $i = \sqrt{-1}$),

$$exp(i\,\theta) = \cos(\theta) + i\,\sin(\theta)$$

Using Euler's result

$$\begin{aligned}
\cos(\theta+\lambda) + i\sin(\theta+\lambda) &= exp(i(\theta+\lambda)) \\
&= exp(i\theta)\,exp(i\lambda) \\
&= (\cos(\theta) + i\,\sin(\theta))(\cos(\lambda) + i\,\sin(\lambda)) \\
&= (\cos(\theta)\cos(\lambda) - \sin(\theta)\sin(\lambda)) + i(\sin(\theta)\cos(\lambda) + \cos(\theta)\sin(\lambda))
\end{aligned}$$

Hence we get

$$\cos(a+b) = \cos a \cos b - \sin a \sin b, \qquad \cos(2a) = \cos^2 a - \sin^2 b$$

and

$$\sin(a+b) = \cos a \sin b + \cos b \sin a, \qquad \sin(2a) = 2\cos a \sin a$$

Stationarity of random phase now follows by using these identities,

$$EX_t = R\,\mathbb{E}\,\cos(\phi t + U) = R\,\mathbb{E}\,(\cos(\phi t)\cos(U) - \sin(\phi t)\sin(U)) = 0$$

since the integral of sine and cosine is zero over a full period is 0,

$$\int_0^{2\pi} \cos u\; du = \int_0^{2\pi} \sin u\; du = 0$$

Similarly, but with a bit more algebra, (expand the cosine terms and collect the terms in the product)

$$\begin{aligned}
\mathrm{Cov}(X_t, X_s) &= R^2\,\mathbb{E}\,(\cos(\phi t + U)\cos(\phi s + U)) \\
&= R^2\,\mathbb{E}\,\big[\cos^2 U \cos(\phi t)\cos(\phi s) + \sin^2 U \sin(\phi t)\sin(\phi s) \\
&\qquad - \cos U \sin U\,(\cos(\phi t)\sin(\phi s) + \cos(\phi s)\sin(\phi t))\big] \\
&= \frac{R^2}{2}\,(\cos\phi t \cos\phi s + \sin\phi t \sin\phi s) \\
&= \frac{R^2}{2}\cos(\phi(t-s))
\end{aligned}$$

using the results that the squared norm of the sine and cosine are

$$\frac{1}{2\pi}\int_0^{2\pi} \cos^2 x\; dx = \frac{1}{2\pi}\int_0^{2\pi} \sin^2 x\; dx = 1/2$$

and the orthogonality property

$$\int_0^{2\pi} \cos x \sin x\; dx = 0$$

Nonlinear Models

Linear process A moving average is a weighted sum of the input series, which we can express as the linear equation $Y = C\,X$.

Nonlinear processes describe a time series that does not simply take a weighted average of the input series. For example, we can allow the weights to depend on the value of the input:

$$Y_t = c_{-1}(X_{t-1}) + c_0(X_t) + c_1(X_{t+1})$$

The conditions that assure stationarity depend on the nature of the input series and the functions $c_j(X_t)$.

Example To form a nonlinear process, simply let prior values of the input sequence determine the weights. For example, consider

$$Y_t = X_t + \alpha X_{t-1} X_{t-2} \tag{2}$$

eBcause the expression for $\{Y_t\}$ is not linear in $\{X_t\}$, the process is nonlinear. Is it stationary?

(Think about this situation: Suppose $\{X_t\}$ consists of iid r.v.s. What linear process does $\{Y_t\}$ resemble? If we were to model such data as this linear process, we would miss a very useful, improved predictor.)

Recursive More interesting examples of nonlinear processes use some type of feedback in which the current value of the process Y_t is determined by past values Y_{t-1}, Y_{t-2}, ... as well as past values of the input series. For example, the following is an example of a *bilinear* process:

$$Y_t = X_t + \alpha_1 Y_{t-1} + \alpha_2 Y_{t-1}\, X_{t-1}$$

Can we recognize the presence of nonlinearity in data?