

## *Examples of Stationary Time Series*

### Overview

1. Stationarity
2. Linear processes
3. Cyclic models
4. Nonlinear models

### Stationarity

**Strict stationarity (Defn 1.6)** Probability distribution of the stochastic process  $\{X_t\}$  is invariant under a shift in time,

$$\begin{aligned} \mathbb{P}(X_{t_1} \leq x_1, X_{t_2} \leq x_2, \dots, X_{t_k} \leq x_k) &= F(x_{t_1}, x_{t_2}, \dots, x_{t_k}) \\ &= F(x_{h+t_1}, x_{h+t_2}, \dots, x_{h+t_k}) \\ &= \mathbb{P}(X_{h+t_1} \leq x_1, X_{h+t_2} \leq x_2, \dots, X_{h+t_k} \leq x_k) \end{aligned}$$

for any time shift  $h$  and  $x_j$ .

**Weak stationarity (Defn 1.7)** (aka, second-order stationarity) The mean and autocovariance of the stochastic process are finite and invariant under a shift in time,

$$\mathbb{E} X_t = \mu_t = \mu \quad \text{Cov}(X_t, X_s) = \mathbb{E}(X_t - \mu_t)(X_s - \mu_s) = \gamma(t, s) = \gamma(t-s)$$

The separation rather than location in time matters.

**Equivalence** If the process is Gaussian with finite second moments, then weak stationarity is equivalent to strong stationarity. Strict stationarity implies weak stationarity only if the necessary moments exist.

**Relevance** Stationarity matters because it provides a framework in which averaging makes sense. Unless properties like the mean and covariance are either fixed or “evolve” in a known manner, we cannot average the observed data.

**What operations produce a stationary process?** Can we recognize/identify these in data?

## Moving Average

**White noise** Sequence of uncorrelated random variables with finite variance,

$$\mathbb{E} W_t = \mu \stackrel{\text{often}}{=} 0 \quad \text{Cov}(W_t, W_s) = \begin{cases} \sigma_w^2 \stackrel{\text{often}}{=} 1 & \text{if } t = s, \\ 0 & \text{otherwise} \end{cases}$$

The input component ( $\{X_t\}$  in what follows) is often modeled as white noise. Strict white noise replaces uncorrelated by independent.

**Moving average** A stochastic process formed by taking a weighted average of another time series, often formed from white noise. If we define  $\{Y_t\}$  from  $\{X_t\}$  as

$$Y_t = \sum_{i=-\infty}^{\infty} c_i X_{t-i}$$

then  $\{Y_t\}$  is a moving average of  $\{X_t\}$ . In order for  $\{Y_t\}$  to have second moments (if the input series  $\{X_t\}$  has second moments), then  $\{c_i\} \in \ell_2$ . ( $\ell_2$  is the space of all square summable sequences, those for which  $\sum c_i^2 < \infty$ .)

### Examples

trivial	$Y_t = X_t$
differences	$Y_t = X_t - X_{t-1}$
3-term	$Y_t = (X_{t+1} + X_t + X_{t-1}) / 3$
one-sided	$Y_t = \sum_{i=0}^{\infty} c_i X_{t-i}$

**Examples in R** The important commands to know are `rnorm` for simulating Gaussian white noise and `filter` to form the filtering. (Use the concatenation function `c` to glue values into a sequence.

Note the practical issue of *end values*: What value do you get for  $y_1$  when forming a moving average? Several approaches are

1. Leave them missing.
2. Extend the input series by, say, back-casting  $x_0, x_1, \dots$  (such as by the mean or fitting a line).
3. Wrapping the coefficient weights (convolution).
4. Reflecting the coefficients at the ends.

**Question:** For what choices of the weights  $c_j$  does the moving average look “smoother” than the input realization?

**Stationarity** of the mean of a moving average  $\{Y_t\}$  is immediate if  $E X_t = \mu_x$  and the sum of the weights is finite. For  $\{Y_t\}$  to be second-order stationary, then (assume that the mean of  $\{X_t\}$  is zero, implying that the mean of  $\{Y_t\}$  is also zero and that the weights are square summable)

$$\gamma_y(t, t+h) = \mathbb{E} Y_t Y_{t+h} = \mathbb{E} \sum_{i,j} c_i c_j X_{t-i} X_{t+h-j} = \sum_{i,j} c_i c_j \gamma_x(t-i, t+h-j)$$

If the input is weakly stationary, then

$$\gamma_y(h) = \sum_{i,j} c_i c_j \gamma_x(t-i, t+h-j) = \sum_{i,j} c_i c_j \gamma_x(h-j+i)$$

If it also happens that the input is white noise, then the covariance further simplifies to a “lagged inner product” of the weights used to construct the moving average,

$$\gamma_y(h) = \sigma_x^2 \sum_i c_i c_{i+h} \tag{1}$$

**Remark** The expression (1) can be thought of as a factorization of the covariance matrix associated with the stochastic process  $\{Y_t\}$ . First, imagine writing  $\{Y_t\}$  as a matrix product

$$Y = C X$$

where the infinite length vectors  $X$  and  $Y$  stack the elements of the two processes

$$X' = (\dots, X_{-2}, X_{-1}, X_0, X_1, X_2, \dots) \text{ and } Y' = (\dots, Y_{-2}, Y_{-1}, Y_0, Y_1, Y_2, \dots)$$

and  $C$  is an infinite-dimensional, square, diagonal matrix with  $c_0$  along the diagonal arranged by stacking and shifting the coefficients as

$$C = \begin{pmatrix} \vdots & & & & & & \\ \dots, c_1, c_0, c_{-1}, c_{-2}, c_{-3} \dots & & & & & & \\ \dots, c_2, c_1, c_0, c_{-1}, c_{-2}, \dots & & & & & & \\ \dots, c_3, c_2, c_1, c_0, c_{-1}, \dots & & & & & & \\ \vdots & & & & & & \end{pmatrix}$$

If the input is white noise, then (1) represents the covariance matrix  $\Gamma$  as the product

$$\Gamma_y = \sigma_x^2 CC'$$

with  $\Gamma$  holding the covariances

$$\Gamma_{y,ij} = \gamma_y(i - j)$$

## Recursive Processes (Autoregression)

**Feedback** Allow past values of the process to influence current values:

$$Y_t = \alpha Y_{t-1} + X_t$$

Usually, the input series in these models would be white noise.

**Stationarity** To see when/if such a process is stationary, use back-substitution to write such a series as a moving average:

$$\begin{aligned} Y_t &= \alpha(\alpha Y_{t-2} + X_{t-1} + X_t) \\ &= \alpha^2(\alpha Y_{t-3} + X_{t-2}) + X_t + \alpha X_{t-1} \\ &= X_t + \alpha X_{t-1} + \alpha^2 X_{t-2} + \dots \end{aligned}$$

Stationarity requires that  $|\alpha| < 1$ . If  $\alpha = 1$ , then you have random walk if  $\{X_t\}$  consists of independent inputs.

**Linear** The ability to represent the autoregression (in this case, a first-order autoregression) as a moving average implies that the autoregression is a linear process (albeit, one with an infinite sequence of weights).

## Cyclic Processes

**Random phase model** Define a stochastic process as follows. Let  $U$  denote a random variable that is uniformly distributed on  $[-\pi, \pi]$ , and define (Here  $R$  is a constant, but we could allow it to be an independent r.v. with mean zero and positive variance.)

$$X_t = R \cos(\phi t + U)$$

Each realization of  $\{X_t\}$  is a single sinusoidal “wave” with frequency  $\phi$  and amplitude  $R$ .

**Trig sums** are always easier in complex notation unless you use them a lot.

You just need to recall Euler's form (with  $i = \sqrt{-1}$ ),

$$\exp(i\theta) = \cos(\theta) + i \sin(\theta)$$

Using Euler's result

$$\begin{aligned} \cos(\theta + \lambda) + i \sin(\theta + \lambda) &= \exp(i(\theta + \lambda)) \\ &= \exp(i\theta) \exp(i\lambda) \\ &= (\cos(\theta) + i \sin(\theta))(\cos(\lambda) + i \sin(\lambda)) \\ &= (\cos(\theta) \cos(\lambda) - \sin(\theta) \sin(\lambda)) + i(\sin(\theta) \cos(\lambda) + \cos(\theta) \sin(\lambda)) \end{aligned}$$

Hence we get

$$\cos(a + b) = \cos a \cos b - \sin a \sin b, \quad \cos(2a) = \cos^2 a - \sin^2 b$$

and

$$\sin(a + b) = \cos a \sin b + \cos b \sin a, \quad \sin(2a) = 2 \cos a \sin a$$

**Stationarity of random phase** now follows by using these identities,

$$EX_t = R \mathbb{E} \cos(\phi t + U) = R \mathbb{E} (\cos(\phi t) \cos(U) - \sin(\phi t) \sin(U)) = 0$$

since the integral of sine and cosine is zero over a full period is 0,

$$\int_0^{2\pi} \cos u \, du = \int_0^{2\pi} \sin u \, du = 0$$

Similarly, but with a bit more algebra, (expand the cosine terms and collect the terms in the product)

$$\begin{aligned} \text{Cov}(X_t, X_s) &= R^2 \mathbb{E} (\cos(\phi t + U) \cos(\phi s + U)) \\ &= R^2 \mathbb{E} [\cos^2 U \cos(\phi t) \cos(\phi s) + \sin^2 U \sin(\phi t) \sin(\phi s) \\ &\quad - \cos U \sin U (\cos(\phi t) \sin(\phi s) + \cos(\phi s) \sin(\phi t))] \\ &= \frac{R^2}{2} (\cos \phi t \cos \phi s + \sin \phi t \sin \phi s) \\ &= \frac{R^2}{2} \cos(\phi(t - s)) \end{aligned}$$

using the results that the squared norm of the sine and cosine are

$$\frac{1}{2\pi} \int_0^{2\pi} \cos^2 x \, dx = \frac{1}{2\pi} \int_0^{2\pi} \sin^2 x \, dx = 1/2$$

and the orthogonality property

$$\int_0^{2\pi} \cos x \sin x \, dx = 0$$

## Nonlinear Models

**Linear process** A moving average is a weighted sum of the input series, which we can express as the linear equation  $Y = C X$ .

**Nonlinear processes** describe a time series that does not simply take a weighted average of the input series. For example, we can allow the weights to depend on the value of the input:

$$Y_t = c_{-1}(X_{t-1}) + c_0(X_t) + c_1(X_{t+1})$$

The conditions that assure stationarity depend on the nature of the input series and the functions  $c_j(X_t)$ .

**Example** To form a nonlinear process, simply let prior values of the input sequence determine the weights. For example, consider

$$Y_t = X_t + \alpha X_{t-1} X_{t-2} \quad (2)$$

Because the expression for  $\{Y_t\}$  is not linear in  $\{X_t\}$ , the process is nonlinear. Is it stationary?

(Think about this situation: Suppose  $\{X_t\}$  consists of iid r.v.s. What linear process does  $\{Y_t\}$  resemble? If we were to model such data as this linear process, we would miss a very useful, improved predictor.)

**Recursive** More interesting examples of nonlinear processes use some type of feedback in which the current value of the process  $Y_t$  is determined by past values  $Y_{t-1}, Y_{t-2}, \dots$  as well as past values of the input series. For example, the following is an example of a *bilinear* process:

$$Y_t = X_t + \alpha_1 Y_{t-1} + \alpha_2 Y_{t-1} X_{t-1}$$

Can we recognize the presence of nonlinearity in data?