Hilbert Spaces

Overview

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3. Hilbert spaces
4. Projection
5. Orthonormal bases
6. Separability and the fundamental isomorphism
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Ideas

Rationale for studying Hilbert spaces:

1. Formalize intuition from regression on prediction, orthogonality
2. Define infinite sums of random variables, such as $\sum_j \psi_j w_{t-j}$
3. Frequency domain analysis
4. Representation theorems

The related Fourier analysis establishes an isometry between the collection of stationary stochastic processes $\{X_t\}$ and squared integrable functions on $[-\pi, \pi]$. This isometry lets us replace

$$X_t \Rightarrow e^{it\lambda}$$

in a way that preserves covariances. In the notation of inner-products, $\langle x, y \rangle$

$$\text{Cov}(X_{t+h}, X_t) = \langle X_{t+h}, X_t \rangle = \langle e^{i(t+h)\lambda}, e^{it\lambda} \rangle_f = \int_{-\pi}^{\pi} e^{ih\lambda} f(\lambda) d\lambda$$
more generally \[ = \int_{-\pi}^{\pi} e^{ih\lambda} dF(\lambda) \]

for a suitably defined function \( f \), the spectral density function, or \( F \), the spectral measure. (The spectral density might not exist.)

**Fourier transform** The relationship \( \gamma(h) = \int_{-\pi}^{\pi} e^{ih\lambda} f(\lambda) d\lambda \) indicates that the s.d.f. is the Fourier transform of the covariances. The Fourier transform is an isometry between Hilbert spaces.

**Related ideas** For more reading see Appendix C in Shumway and Stoffer as well as these classics (and one newer edition)


and for time series, see Brockwell and Davis, *Time Series: Theory and Methods*.

**Vector Spaces**

**Key ideas** associated with a vector space (a.k.a, linear space) are subspaces, basis, and dimension.

**Define.** A complex vector space is a set \( \mathcal{V} \) of elements called vectors that satisfy the following axioms on addition of vectors

1. For every \( x, y \in \mathcal{V} \), there exists a sum \( x + y \in \mathcal{V} \)
2. Addition is commutative, \( x + y = y + x \).
3. Addition is associative, \( x + (y + z) = (x + y) + z \).
4. There exists an origin denoted \( 0 \) such that \( x + 0 = x \).
5. For every \( x \in \mathcal{V} \), there exists \( -x \) such that \( x + (-x) = 0 \), the origin.

and the following axioms on multiplication by a (complex) scalar

1. For every \( x \in \mathcal{V} \) and \( \alpha \in \mathbb{C} \), the product \( \alpha x \in \mathcal{V} \).
2. Multiplication is commutative.
3. 1 \times x = x.
4. Multiplication is distributive: \( \alpha(x+y) = \alpha x + \alpha y \) and \( (\alpha+\beta)x = \alpha x + \beta x \).

A vector space must have at least one element, the origin.

**Examples.** Some common examples of vector spaces are:

1. Set of all complex numbers \( \mathbb{C} \) or real numbers \( \mathbb{R} \).
2. Set of all polynomials with complex coefficients.
3. Euclidean \( n \)-space ("vectors") whose elements are complex numbers, often labelled \( \mathbb{C}^n \).

**Subspaces.** A set \( \mathcal{M} \subset \mathcal{V} \) is a subspace or linear manifold if it is algebraically closed,

\[
x, y \in \mathcal{M} \implies \alpha x + \beta y \in \mathcal{M}.
\]

Consequently, each subspace must include the origin, since \( x - x = 0 \). The typical way to generate a subspace is to begin with a collection of vectors and consider the set of all possible linear combinations of this set; the resulting collection is a subspace. Intersections of subspaces are also subspaces. (Note: "closure" here is not in the sense of open and closed sets. A vector space need not have a topology.)

**Linear dependence.** A countable set of vectors \( \{x_i\} \) is linearly dependent if there exists a set of scalars, not all zero, s.t.

\[
\sum_i \alpha_i x_i = 0.
\]

The sum \( \sum_i \alpha_i x_i \) is known as a linear combination of vectors. Alternatively, the collection of vectors \( \{x_i\} \) is linearly dependent iff some member \( x_k \) is a linear combination the preceding vectors.

**Bases and dimension.** A basis for \( \mathcal{V} \) is a set of linear independent vectors \( \{x_i\} \) such that every vector \( v \in \mathcal{V} \) can be written as a linear combination of the basis,

\[
v = \sum_i \alpha_i x_i.
\]
The *dimension* of a vector space is the number of elements in a basis for that space.

**Isometry.** Two vector spaces are *isomorphic* if there exists a linear bijection (one-to-one, onto) \( T : X \rightarrow Y \),

\[
T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2).
\]

**Metric spaces**

**Distance** metric defines topology (open, closed sets) and convergence.

**Define.** A metric space \( X \) combines a set of elements (that need not be a vector space) with the notion of a distance, called the *metric*, \( d \). The metric \( d : (X \times X) \rightarrow \mathbb{R} \) must satisfy:

- Non-negative: \( d(x, y) \geq 0 \), with equality iff \( x = y \).
- Symmetric: \( d(x, y) = d(y, x) \).
- Triangle: \( d(x, y) \leq d(x, z) + d(z, y) \).

**Examples.** Three important metrics defined on space of continuous functions \( C[a, b] \) are

- \( d_\infty(f, g) = \max |f(x) - g(x)| \). (Uniform topology)
- \( d_1(f, g) = \int_a^b |f(x) - g(x)| \, dx \).
- \( d_2(f, g)^2 = \int_a^b (f(x) - g(x))^2 \, dx \).

**Convergence** is defined in the metric, \( x_n \rightarrow x \) if \( d(x_n, x) \rightarrow 0 \). Different metrics induce different notions of convergence. The “triangle functions” \( h_n \) defined on \( \frac{1}{2n+1}, \frac{1}{2n}, \frac{1}{2n-1} \) converge to zero in \( d_1 \), but not in \( d_\infty \).

**Cauchy sequences and completeness.** The sequence \( x_n \) is Cauchy if for all \( \epsilon > 0 \), there exists an \( N \) such that \( n, m \geq N \) implies \( d(x_n, x_m) < \epsilon \). All convergent sequences must be Cauchy, though the converse need not be true. If all Cauchy sequences converge to a member of the metric space, the space is said to be *complete*. 
Since the triangle functions have disjoint support, \( d_\infty(h_n, h_m) = 1 \), \( \{h_n\} \) is not Cauchy, and thus does not converge. With \( d_\infty \), \( C[a, b] \) is complete since sequences like \( \{h_n\} \) do not converge in this metric; \( C[a, b] \) is not complete with \( d_1 \) (or \( d_2 \)) as the metric.

**Continuous functions.** With the notion of convergence in hand, we can define a function \( f \) to be continuous if it preserves convergence of arguments. The function \( f \) is continuous iff

\[
x_n \to x \implies f(x_n) \to f(x).
\]

**Isometry.** Two metric spaces \((X, d_x)\) and \((Y, d_y)\) are isometric if if there exists a bijection (1-1, onto) \( f : X \to Y \) which preserves distance,

\[
d_x(a, b) = d_y(f(a), f(b)).
\]

Isomorphism between two vector spaces requires the preservation of linearity (linear combinations), whereas in metric spaces, the metric must preserve distance. These two notions — the algebra of vector spaces and distances of metric spaces — combine in normed linear spaces.

**Normed linear spaces.**

**Combine** the algebra of vector spaces and distance of metric spaces.

**Define.** A normed vector space \( V \) is a vector space together with a real-valued function \( \|x\| \), the “norm” which is

1. Non-negative: \( \|x\| \geq 0 \), with equality iff \( x = 0 \).
2. Scalar mult: \( \|\alpha x\| = |\alpha| \|x\| \).
3. Triangle: \( \|x + y\| \leq \|x\| + \|y\| \).

**Continuity of norm.** If \( x_n \to x \), then \( \|x_n\| \to \|x\| \). This follows from the triangle inequality (noting \( x = x - x_n + x_n \))

\[
\|\|x_n\| - \|x\|\| \leq \|x_n - x\|.
\]
Complete space A normed vector space $V$ is complete if all Cauchy sequences $\{X_i\} \in V$ have limits within the space:

$$\lim \|X_i - X_j\| \to 0 \implies \lim X_i \in V$$

Examples. Several common normed spaces are $\ell_1$ and the collection $L^1$ of Lebesgue integrable functions with the norm

$$\|f\| = \int_{-\infty}^{\infty} |f(x)|dx < \infty.$$

While it is true that $\ell_1 \subset \ell_2$, the same does not hold for functions due to problems at the origin. There is not a nesting of $L^2$ and $L^1$. For example,

$$1/(1 + |x|) \in L^2 \text{ but not in } L^1.$$

Conversely,

$$|x|^{-1/2}e^{-|x|} \in L^1 \text{ but not in } L^2.$$

Remarks. Using the Lebesgue integral, $L^1$ is complete and is thus a Banach space. Also, the space of continuous functions $C[a, b]$ is dense is $L^1[a, b]$.

Hilbert Spaces

Geometry Hilbert spaces conform to our sense of geometry and regression. For example, a key notion is the orthogonal decomposition (data = fit + residual, or $Y = \hat{Y} + (Y - \hat{Y})$).

Inner product space A vector space $H$ is an inner-product space if for each $x, y \in H$ there exists a real-valued, bilinear function $\langle x, y \rangle$ which is

1. Linear: $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
2. Scalar multiplication: $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$
3. Non-negative: $\langle x, x \rangle \geq 0$, with $\langle x, x \rangle = 0$ iff $x = 0$
4. Conjugate symmetric: $\langle x, y \rangle = \overline{\langle y, x \rangle}$ (symmetry in real-valued case)
**Hilbert space**  A Hilbert space is a *complete* inner-product space. An inner-product space can always be “completed” to a Hilbert space by adding the limits of its Cauchy sequences to the space.

**Examples** The most common examples of Hilbert spaces are

1. Euclidean \(\mathbb{R}^n\) and \(\mathbb{C}^n\) with inner products defined by the dot-product \(\langle x, y \rangle = \sum_i x_i \bar{y}_i\).
2. \(\ell_2\) sequences (square summable sequences). This is the canonical Hilbert space.
3. \(L_2[a,b]; f \in L_2\) iff \(\int_a^b f^2 < \infty\). \(L_2\) is complete, and is thus a Hilbert space. Note that the inner-product \(\langle f, g \rangle = \int f \bar{g}\) is valid (integrable) since

   \[
   (f - g)^2 \geq 0 \implies |f(x)\bar{g(x)}| \leq (|f(x)|^2 + |g(x)|^2)/2
   \]

   so that the product \(f \bar{g}\) is integrable (lies in \(L_1\)).
4. Random variables with finite variance, an idea that we will explore further. The inner product is \(\langle X, Y \rangle = \text{Cov}(X, Y)\).

**Norm**  Every Hilbert space has an associated *norm* defined using its inner product,

\[
\|x\|^2 = \langle x, x \rangle,
\]

which reduces to the (squared) length of a vector in \(\mathbb{R}^n\). Observe that \(\|\alpha x\| = |\alpha| \|x\|\), as in the definition of a normed space. Norms in i.p. spaces are special; in particular, they also satisfy the Parallelogram Law

\[
\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)
\]

**Orthonormal**  If \(\langle x, y \rangle = 0\), then \(x\) and \(y\) are orthogonal, often written as \(x \perp y\). A collection of orthogonal vectors having norm 1 is an *orthonormal* set. For example, in \(\mathbb{R}^n\), the columns of an orthogonal matrix form an orthonormal set.

**Pythagorean theorem**  Let \(\mathcal{X} = \{x_j\}_{j=1}^n\) denote an orthonormal set in \(\mathcal{H}\). Then for any \(x \in \mathcal{H}\),

\[
\|x\|^2 = \sum_{j=1}^n |\langle x, x_j \rangle|^2 + \|x - \sum_j \langle x, x_j \rangle x_j\|^2.
\]  \(\text{ (1)}\)
We know this in statistics as the ANOVA decomposition in statistics, \( \text{Total SS} = \text{Fit SS} + \text{Resid SS} \). Furthermore, the vector \( r = x - \sum_i \langle x, x_i \rangle x_i \) is orthogonal to the subspace spanned by \( \mathcal{X} \). Compare to

\[
x = (I - H + H)x = Hx + (I - H)x
\]

where \( H \) is a projection (idempotent) matrix.

**Proof.** Begin with the identity (once again, add and subtract)

\[
x = \sum_j \langle x, x_j \rangle x_j + \left( x - \sum_j \langle x, x_j \rangle x_j \right),
\]

The two on the r.h.s are orthogonal and thus (1) holds. The coefficients \( \langle x, x_j \rangle \) seen in (1) are known as Fourier coefficients.

**Bessel's inequality**  If \( \mathcal{X} = \{x_1, \ldots, x_n\} \) is an orthonormal set in \( \mathcal{H} \), \( x \in \mathcal{H} \) is any vector, and the Fourier coefficients are \( \alpha_j = \langle x, x_j \rangle \), then

\[
\|x\|^2 \geq \sum_j |\alpha_j|^2.
\]

**Proof.** Immediate from Pythagorean theorem.

**Cauchy-Schwarz inequality**  For any \( x, y \in \mathcal{H} \),

\[
|\langle x, y \rangle| \leq \|x\| \|y\|.
\]

Equality occurs when \( \{x, y\} \) are linearly dependent. Hence we can think of the norm as an upper bound on the size of inner products:

\[
\|x\| = \max_{\|y\|=1} |\langle x, y \rangle|.
\]

**Proof.** The proof suggests that the C-S inequality is closely related to the ideas of projection. The result is immediate if \( y = 0 \). Assume \( y \neq 0 \) and consider the orthonormal set \( \{y/\|y\|\} \). Bessel’s inequality implies

\[
|\alpha|^2 = \langle x, y/\|y\| \rangle^2 \leq \|x\|^2.
\]

Equality occurs when \( x \) is a multiple of \( y \), for then the term omitted from the Pythagorean theorem that leads to Bessel’s inequality is zero.
Some results that are simple to prove with the Cauchy-Schwarz theorem are:

1. The inner product is continuous, \( \langle x_n, y_n \rangle \to \langle x, y \rangle \). The proof essentially uses \( x_n = x_n - x + x \) and the Cauchy-Schwarz theorem. Thus, we can always replace \( \langle x, y \rangle = \lim_n \langle x_n, y \rangle \).

2. Every inner product space is a normed linear space, as can be seen by using the C-S inequality to verify the triangle inequality for the implied norm.

Isometry between Hilbert spaces combines linearity from vector spaces with distances from metric spaces. Two Hilbert spaces \( H_1 \) and \( H_2 \) are isomorphic if there exists a linear function \( U \) which preserves inner products,

\[
\forall x, y \in H_1, \quad \langle x, y \rangle_1 = \langle Ux, Uy \rangle_2.
\]

Such an operator \( U \) is called unitary. The canonical example of an isometry is the linear transformation implied by an orthogonal matrix.

Summary of theorems: For these, \( \{x_j\}_{j=1}^n \) denotes a finite orthonormal set in an inner product space \( H \):

1. Pythagorean theorem: \( \|x\|^2 = \sum_{j=1}^n |\langle x, x_j \rangle|^2 + \|x - \sum_j \langle x, x_j, x \rangle_j \|^2 \).
2. Bessel’s inequality: \( \|x\|^2 \geq \sum_j |\langle x, x_j \rangle|^2 \).
3. Cauchy-Schwarz inequality: \( |\langle x, y \rangle| \leq \|x\| \|y\| \).
4. Inner product spaces are normed spaces with \( \|x\|^2 = \langle x, x \rangle \). This norm satisfies the parallelogram law.
5. The i.p. is continuous, \( \lim_n \langle x_n, y \rangle = \langle x, y \rangle \).

Projection

Orthogonal complement: Let \( M \) denote any subset of \( H \). Then the set of all vectors orthogonal to \( M \) is denoted \( M^\perp \), meaning

\[
x \in M, y \in M^\perp \quad \Rightarrow \quad \langle x, y \rangle = 0.\]

Notice that
1. \( \mathcal{M}^\perp \) is a subspace since the i.p. is linear.

2. \( \mathcal{M}^\perp \) is closed (contains limit points) since the i.p. is continuous:

\[ y \in \mathcal{M}, x_n \in \mathcal{M}^\perp \rightarrow x = \lim_n \langle x_n, y \rangle = 0 \]

**Projection lemma**  Let \( \mathcal{M} \) denote a closed subspace of \( \mathcal{H} \). Then for any \( x \in \mathcal{H} \), there exists a unique element \( \hat{x} = \mathcal{P}_\mathcal{M}x \in \mathcal{M} \) closest to \( x \),

\[ d = \inf_{y \in \mathcal{M}} \| x - y \|^2 = \| x - \hat{x} \|^2, \quad \hat{x} \text{ is unique.} \]

The vector \( \hat{x} \) is known as the projection of \( x \) onto \( \mathcal{M} \). A picture suggests the “shape” of closed subspaces in a Hilbert space is very regular (not “curved”). This lemma only says that such a closest element exists; it does not attempt to describe it.

**Proof.** Relies on the parallelogram law and closure properties of the subspace. The first part of the proof shows that there is a Cauchy sequence \( y_n \) in \( \mathcal{M} \) for which \( \lim \| x - y_n \| = \inf_\mathcal{M} \| x - y \| \). To see unique, suppose there were two, then use the parallelogram law to show that they are the same:

\[
0 \leq \| \hat{x} - \hat{z} \|^2 = \| (\hat{x} - x) - (\hat{z} - x) \|^2 = -\| (\hat{x} - x) + (\hat{z} - x) \|^2 + 2(\| \hat{x} - x \|^2 + \| \hat{z} - x \|^2) = -4\| (x - \hat{x})/2 - x \|^2 + 2(\| \hat{x} - x \|^2 + \| \hat{z} - x \|^2) \leq -4d + 4d = 0
\]

**Projection theorem.** Let \( \mathcal{M} \) denote a closed subspace of \( \mathcal{H} \). Then every \( x \in \mathcal{M} \) can be uniquely written as

\[ x = \mathcal{P}_\mathcal{M}x + z \text{ where } z \in \mathcal{M}^\perp \]

**Proof.** Let \( \mathcal{P}_\mathcal{M}x \) be the vector identified in the lemma so that uniqueness is established. Define \( z = x - \mathcal{P}_\mathcal{M}x \). The challenge is to show that \( \langle z, y \rangle = 0 \) for all \( y \in \mathcal{M} \) so that \( z \) indeed lies in \( \mathcal{M}^\perp \). Again, the proof is via a contradiction. Suppose \( \exists y \in \mathcal{M} \) such that \( \langle x - \hat{x}, y \rangle \neq 0 \). This contradicts \( \hat{x} \) being the closest to \( x \). Let \( b = \langle x - \hat{x}, y \rangle/\| y \|^2 \), the “regression coefficient of the residual \( x - \hat{x} \) on \( y \). Using real numbers,

\[
\| x - \hat{x} - by \|^2 = \| x - \hat{x} \|^2 - \frac{\langle x - \hat{x}, y \rangle^2}{\| y \|^2} < \| x - \hat{x} \|^2
\]

a contradiction.
Properties of projection mapping  Important properties of the projection mapping $P_M$ are

1. Linear: $P_M(\alpha x + \beta y) = \alpha P_M x + \beta P_M y$.
2. Anova decomposition: $\|x\|^2 = \|P_M x\|^2 + \|(I - P_M) x\|^2$.
3. Representation $x = P_M x + (I - P_M) x$ is unique from the projection theorem.
4. Continuous: $P_M x_n \to P_M x$ if $x_n \to x$. (use linearity and the anova decomposition)
5. Idempotent: $P_M x = x \iff x \in M$ and $P_M x = 0 \iff x \in M^\perp$
6. Subspaces: $P_{M_1} P_{M_2} x = P_{M_1} x \iff M_1 \subseteq M_2$.

Regression  Least squares regression fits nicely into the Hilbert space setting. Let $H$ denote real Euclidean n-space $R^n$ with the usual dot-product as inner product, and let $M$ denote the subspace formed by linear combinations of the vectors $x_1, x_2, \ldots, x_k$.

Consider a vector $y \in H$. The projection theorem tells us that we can form an orthogonal decomposition of $y$ as

$$y = P_X y + z \text{ where } P_X y = \sum \alpha_j x_j,$$

and $z = y - P_X y$. Since $\langle z, x_j \rangle = 0$, we obtain a system of equations (the normal equations — it’s also clear now why these are called the normal equations!)

$$\langle z, x_j \rangle = \langle y - \sum \alpha_i x_i, x_j \rangle = 0, \quad j = 1, \ldots, k$$

Solving this system gives the usual OLS regression coefficients. Notice that we can also express the projection theorem explicitly as

$$y = H y + (I - H) y,$$

where the idempotent projection matrix $P_X = H$ is $H = X (X' X)^{-1} X'$, the “hat matrix”.
Orthonormal bases.

Regression is most easy to interpret and compute if the columns \( x_1, x_2, \ldots, x_k \) are orthonormal. In that case, the normal equations are diagonal and regression coefficients are simply \( \alpha_j = \langle y, x_j \rangle \). This idea of an orthonormal basis extends to all Hilbert spaces, not just those that are finite dimensional. If the o.n. basis \( \{x_j\}_{j=1}^n \) is finite, though, the projection is \( P_M y = \sum \langle y, x_j \rangle x_j \) as in regression with orthogonal \( X \).

**Theorem.** Every Hilbert space has an orthonormal basis.

The proof amounts to Zorn’s lemma or the axiom of choice. Consider the collection of all orthonormal sets, ...

**Fourier representation** Let \( \mathcal{H} \) denote a Hilbert space and let \( \{x_\alpha\} \) denote an orthonormal basis (Note: \( \alpha \) is a member of some set \( A \), not just integers.) Then for any \( y \in \mathcal{H} \), we have

\[
y = \sum_A \langle y, x_\alpha \rangle x_\alpha \quad \text{and} \quad \|y\|^2 = \sum_A |\langle y, x_\alpha \rangle|^2
\]

The latter equality is called Parseval’s identity.

**Proof.** Bessel’s inequality works for half of the equality for any finite subsets \( A' \subset A \),

\[
\sum_{A'} |\langle y, x_\alpha \rangle|^2 \leq \|y\|^2.
\]

This implies that \( \langle y, x_\alpha \rangle \neq 0 \) for at most countable \( \alpha \)'s so that (with some ordering of the elements of \( A, j = \alpha_j \)) \( \sum_{j=1}^n |\langle y, x_j \rangle|^2 \) is a monotone series with an upper bound and is thus convergent as \( n \to \infty \). The proof continues by showing that the resulting approximation \( \hat{y}_n = \sum_{j=1}^n \langle y, x_j \rangle x_j \) converges to \( y \).

Now show it’s Cauchy, and use completeness of \( \mathcal{H} \) to conclude that the limit \( y' \) must be \( y \),

\[
\langle y - y', x_k \rangle = \lim_n \langle y - \sum_{j=1}^n \langle y, x_j \rangle x_j, x_k \rangle = \langle y, x_k \rangle - \langle y, x_k \rangle = 0.
\]
For any other \( \alpha \neq \alpha_j \), the same argument shows \( \langle y - y', x_{\alpha} \rangle = 0 \). Since \( y - y' \) is orthogonal to all of the \( x_{\alpha} \)'s, it must be zero (or we could extend the orthonormal basis).

To prove the norm relationship, use the continuity of the norm and orthogonality,

\[
0 = \lim_n \|y - \sum_{j=1}^n \langle y, x_j \rangle x_j \|^2 = \|y\|^2 - \sum_A |\langle y, x_{\alpha} \rangle|^2
\]

**Construction**  The *Gram-Schmidt* construction converts a set of vectors into an orthonormal basis. The method proceeds recursively,

\[
\begin{align*}
x_1 &\Rightarrow o_1 = x_1/\|x_1\| \\
x_2 &\Rightarrow u_2 = x_2 - \langle x_2, o_1 \rangle o_1, o_2 = u_2/\|u_2\| \\
\cdots \\
x_n &\Rightarrow u_n = x_n - \sum_{j=1}^{n-1} \langle x_n, o_j \rangle o_j, o_n = u_n/\|u_n\|
\end{align*}
\]

**QR decomposition**  In regression analysis, a modified version of the Gram-Schmidt process leads to the so-called QR decomposition of the matrix \( X \). The QR decomposition expresses the covariate matrix \( X \) as

\[
X = QR \text{ where } Q'Q = I,
\]

and \( R \) is upper-triangular. With \( X \) in this form, one solves the modified system

\[
Y = X\beta + \epsilon \Rightarrow Y = Q(\alpha = R\beta) + \epsilon
\]

using \( \hat{\alpha}_j = \langle Y, q_j \rangle \). The \( \beta \)'s come via back-substitution if needed.

**Separability and the Fundamental Isomorphism.**

**Separable**  A Hilbert space is *separable* if it has a countable dense subset. Examples: (1) real number system (rationals), (2) Continuous functions \( C[a,b] \) (polynomials with rational coefs). A Hilbert space is separable if it has a countable orthonormal basis.

**Proof.** If its separable, use G-S to convert the countable dense subset to an orthonormal set (removing those that are dependent). If it has a countable basis, use the Fourier representation to see that it is dense.
**Isomorphisms**  If a separable Hilbert space is finite dimensional, it is isomorphic to $\mathbb{C}^n$. If it not finite dimensional, it is isomorphic to $\ell_2$.

*Proof.* Define the isomorphism that maps $y \in \mathcal{H}$ to $\ell_2$ by

$$Uy = \{\langle y, x_j \rangle\}_{j=1}^\infty$$

where $\{x_j\}$ is an orthonormal basis. The sequence in $\ell_2$ is the sequence of Fourier coefficients in the chosen basis. Note that the inner product is preserved since

$$\langle y, w \rangle = \langle \sum_j \langle y, x_j \rangle x_j, \sum_k \langle w, x_k \rangle x_k \rangle = \sum_j \langle y, x_j \rangle \overline{\langle w, x_j \rangle}$$

which is the i.p. on $\ell_2$.

**$L_2$ Space of Random variables**

*Define* the inner product space of random variables with finite variance $L_2 = L_2(\Omega, F, P)$ as the collection of measurable complex-valued functions $f$ for which

$$\int f^2(\omega)P(d\omega) = \int f^2 dP < \infty.$$  

With the inner product $\langle f, g \rangle = \int f \overline{g} dP$, $L_2$ is a Hilbert space.

*Translated* to the language of random variables, we form an i.p. space from random variables $X$ for which $EX^2 < \infty$ with the inner product

$$\langle X, Y \rangle = EXY$$

If the random variables have mean zero, then $\langle X, Y \rangle = \text{Cov}(X,Y)$.

**Equivalence classes**  Observe that $\langle X, X \rangle = EX^2 = 0$ does not imply that $X$ is identically zero. It only implies that $X = 0$ a.e. In $L_2$, the symbol $X$ really stands for an *equivalence class* of functions which are equal almost everywhere. The inner product retains the important property that $\langle X, X \rangle = 0$ iff $X = 0$, but the claim only holds for $X$ a.e.
Mean square convergence  Convergence in $L_2$ is convergence in mean square (m.s.),

$$X_n \to X \iff \|X_n - X\| \to 0.$$  

That is, $E(X_n - X)^2$ must go to zero.

Properties  of mean square convergence derive from those of the associated inner product. We can interchange limits with means, variances and covariances. If $\|X_n - X\| \to 0$, then

1. Mean: $\lim_n E(X_n) = \lim_n \langle X_n, 1 \rangle = \langle \lim_n X_n, 1 \rangle = EX$.  
2. Variance: $\lim_n E(X_n^2) = \lim_n \langle X_n, X_n \rangle = \langle X, X \rangle = EX^2$.  
3. Covariance: $\lim_n E X_n Y_n = \lim_n \langle X_n, Y_n \rangle = \langle X, Y \rangle = EXY$

The first two are consequences of the third, with $Y_n = 1$ or $Y_n = X_n$.

Note: Probabilistic modes of convergence are:

- Convergence in probability: $\lim_n P\{\omega : |X_n(\omega) - X(\omega)| < \epsilon\} = 1$.
- Convergence almost surely: $P\{\omega : \lim_n X_n(\omega) = X\} = 1$ or $\lim_n P\{\omega : \sup_{m>n} |X_m(\omega) - X(\omega)| < \epsilon\} = 1$.

Chebyshev’s inequality implies that convergence in mean square implies convergence in probability; also, by definition, a.s. convergence implies convergence in probability. The reverse holds for subsequences. For example, the Borel-Cantelli lemma implies that if a sequence converges in probability, then a subsequence converges almost everywhere. Counter-examples to converses include the “rotating functions” $X_n = I_{[j-1]/k, j/k]}$ and “thin peaks” $X_n = nI_{[0,1/n]}$. I will emphasize mean square convergence, a Hilbert space idea. However, m.s. convergence also implies a.s. convergence along a subsequence.

Projection and conditional expectation

Conditional mean  is the minimum mean squared predictor of any random variable $Y$ given a collection $\{X_1, \ldots, X_n\}$ is the conditional expectation of $Y$ given the $X$’s. Need to assume that $\text{Var}(Y) < \infty$. 
**Proof.** We need to show that for any function \( g \) (not just linear)
\[
\min_g E (Y - g(X_1, \ldots, X_n))^2 = E (Y - E[Y|X_1, \ldots, X_n])^2.
\]
As usual, one cleverly adds and substracts, writing (with \( X \) for \( \{X_1, \ldots, X_n\} \))
\[
E (Y - g(X))^2 = E (Y \pm E[Y|X] - g(X))^2
\]
\[
= E (Y - E[Y|X])^2 + E (E[Y|X] - g(X))^2 + 2E[(E[Y|X] - g(X))E(Y-E[Y|X])]
\]
\[
= E (Y - E[Y|X])^2 + E (E[Y|X] - g(X))^2
\]
\[
> E (Y - E[Y|X])^2
\]

**Projection** The last step in this proof suggests that we can think of the conditional mean as a projection into a subspace. Let \( \mathcal{M} \) denote the closed subspace associated with the \( X \)'s, where by closed we mean random variables \( Z \) that can be expressed as functions of the \( X \)'s. Define a “projection” into \( \mathcal{M} \) as
\[
P_\mathcal{M} Y = E[Y\{X_1, \ldots, X_j, \ldots\}].
\]
This operation has the properties seen for projection in a Hilbert space,
1. Linear \( \langle E[ay + bx|Z] = aE[Y|Z] + bE[X|Z] \rangle \)
2. Continuous \( Y_n \to Y \) implies \( P_\mathcal{M} Y_n \to P_\mathcal{M} Y \).
Indeed, we also obtain a form of orthogonality in that we can write
\[
Y = Y \pm E[Y|X] = E[Y|X] + (Y - E[Y|X])
\]
with
\[
\langle E[Y|X], Y - E[Y|X] \rangle = 0.
\]
Since \( E[Y|nothing] = EY \), the subspace \( \mathcal{M} \) should contain the constant vector 1 for this sense of projection to be consistent with our earlier definitions.

**Tie to regression** The fitted values in regression (with a constant) preserve the covariances with the predictors,
\[
\text{Cov}(Y, X_j) = \text{Cov}(Y \pm \hat{Y}, X_j) = \text{Cov}(\hat{Y}, X_j) \text{ .}
\]
Similarly, for any \( Z = g(X_1, \ldots) \in \mathcal{M}, \)
\[
E[YZ] = E[(Y \pm E[Y|X])Z] = E[E[Y|X]Z]. \tag{2}
\]
Best linear prediction

Linear projection. We need to make it easier to satisfy the orthogonality conditions. Simplest way to do this is to project onto a space formed by linear operations rather than any measurable function. Consider the projection defined as

\[ P_{\mathcal{S}(1, X_1, \ldots, X_n)} Y = \sum_{j=0}^{n} \alpha_j X_j, \quad X_0 = 1, \]

where the coefficients are chosen as in regression to make the “residual” orthogonal to the \(X\)'s; that is, the coefficients satisfy the normal equations

\[ \langle Y, X_k \rangle = \langle \sum_j \alpha_j X_j, X_k \rangle \implies \langle Y - \sum_j \alpha_j X_j, X_k \rangle = 0, \quad k = 0, 1, \ldots, n. \]

Note that

- The m.s.e. of the linear projection will be at least as large as that of the conditional mean, and sometimes much more (see below).
- The two are the same if the random variables are Gaussian.

Example Define \(Y = X^2 + Z\) where \(X, Z \sim N(0, 1)\), and are independent. In this case, \(E[Y|X] = X^2\) which has m.s.e 1. In contrast, the best linear predictor into \(\mathcal{S}(1, X)\) is the combination \(b_0 + b_1 X\) with, from the normal equations,

\[ \langle Y, 1 \rangle = 1 = \langle b_0 + b_1 X, 1 \rangle \]
\[ \langle Y, X \rangle = 0 = \langle b_0 + b_1 X, X \rangle , \]

\(b_0 = 1\) and \(b_1 = 0\). The m.s.e of this predictor is \(E(Y-1)^2 = EY^2 - 1 = 3\).

Predictors for ARMA processes

Infinite past In these examples, the Hilbert space is defined by a stationary process \(\{X_t\}\). We wish to project members of this space into the closed subspace defined by the process up to time \(n\), \(\mathcal{X}_n = \mathcal{S}_p\{X_n, X_{n-1}, \ldots\}\).
AR(p) Let \( \{X_t\} \) denote the covariance stationary AR\( (p) \) process

\[
X_t = \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + w_t
\]

where \( w_t \sim WN(0, \sigma^2) \). What is the best linear predictor of \( X_{n+1} \) in \( \mathcal{X}_n \)? The prediction/orthogonality equations that the predictor \( \hat{X}_{n+1} \) must satisfy are

\[
\langle \hat{X}_{n+1}, X_k \rangle = \langle X_{n+1}, X_k \rangle, \quad k = n, n-1, \ldots
\]

Since \( w_{n+1} \perp \mathcal{X}_n \) we have

\[
\langle X_{n+1}, X_j \rangle = \langle w_t + \sum_{j=1}^p \phi_j X_{t-j}, X_k \rangle = \langle \sum_{j=1}^p \phi_j X_{t-j}, X_k \rangle,
\]

so that \( \hat{X}_{n+1} = \sum_{j=1}^p \phi_j X_{t-j} \). These lead back to the Yule-Walker equations. Note that this argument does not require that the order of the autoregression \( p \) be finite.

MA(1) Let \( \{X_t\} \) denote the invertible MA\( (1) \) process

\[
X_t = w_t - \theta w_{t-1}
\]

with \( |\theta| < 1 \) and \( w_t \sim WN(0, \sigma^2) \). Since the process is invertible, express it as the autoregression \( (1 - \theta B)^{-1} X_t = w_t \), or

\[
X_t = w_t - \theta X_{t-1} - \theta^2 X_{t-2} - \ldots
\]

From the AR example, it follows that \( \hat{X}_{n+1} = -\sum_{j=1}^\infty \theta^j X_{n-j+1} \).

Role for Kalman filter? What about conditioning on the finite past? That’s what the Kalman filter is all about.