Functions Related to Spectral Estimation

- Spectral density of ARMA process

(Mac) In[1]:= \( \phi[z_] := 1 - 1.3 z + 0.95 z^2 \)

(Mac) In[2]:= \( \text{zeros} = \text{Solve}[\phi[z] = 0, z] \)

(Mac) Out[2]= \{ \{z \to 0.684211 - 0.764518 \text{i}\}, \{z \to 0.684211 + 0.764518 \text{i}\} \}

The syntax (expression) /.(rules) uses the rules to replace symbols in the expression.

(Mac) In[3]:= Abs[z /. zeros]
\( \text{Arg}[z /. \text{zeros}] / (2 \pi) \)

(Mac) Out[3]= \{1.02598, 1.02598\}

(Mac) Out[4]= \{-0.133813, 0.133813\}

The product of \( \phi(z) \) times its conjugate is a polynomial in cosine of the frequency \( \lambda \). (Chop removes "near zero" values that occur when manipulating floating point numbers.)

(Mac) In[5]:= \( \phi[e^{2 \pi i \lambda}] \phi[e^{-2 \pi i \lambda}] \) // ExpToTrig
\( \text{FullSimplify}[\%] \)

(Mac) Out[5]= \(\frac{(1 - 1.3 \cos(2 \pi \lambda) + 0.95 \cos(4 \pi \lambda) + 1.3 \sin(2 \pi \lambda) - 0.95 i \sin(4 \pi \lambda)\big)}{(1 - 1.3 \cos(2 \pi \lambda) + 0.95 \cos(4 \pi \lambda) - 1.3 i \sin(2 \pi \lambda) + 0.95 i \sin(4 \pi \lambda)\big)}\)

(Mac) Out[6]= 3.5925 - 5.07 \cos(2 \pi \lambda) + 1.9 \cos(4 \pi \lambda) + 8.88178 \times 10^{-16} i \sin(2 \pi \lambda)

(Mac) In[7]:= \( \phi[e^{2 \pi i \lambda}] \phi[e^{-2 \pi i \lambda}] \) // FullSimplify // Chop

(Mac) Out[7]= 3.5925 - 5.07 \cos(2 \pi \lambda) + 1.9 \cos(4 \pi \lambda)

(Mac) In[8]:= \( \sigma[\lambda_] = \frac{1}{\text{Chop}[\text{FullSimplify}[\phi[e^{2 \pi i \lambda}] \phi[e^{-2 \pi i \lambda}]]]} \)

(Mac) Out[8]= \(\frac{1}{3.5925 - 5.07 \cos(2 \pi \lambda) + 1.9 \cos(4 \pi \lambda)}\)

Mathematica cuts off the sharp peak, so show it on a log scale.
Associated function over the complex plane
\begin{align*}
\text{(Mac) In[11]} &= \text{Plot3D} \left[ \text{Log} \left[ \text{Abs} \left( \frac{1}{\phi[x + i y] \phi[x - i y]} \right) \right], \{x, -1.5, 1.5\}, \{y, -1.5, 1.5\} \right] \\
\text{(Mac) Out[11]} &= \\
\text{(Mac) In[12]} &= \text{ContourPlot} \left[ \text{Log} \left[ \text{Abs} \left( \frac{1}{\phi[x + i y] \phi[x - i y]} \right) \right], \{x, -1.5, 1.5\}, \{y, -1.5, 1.5\}, \text{Epilog} \to \text{Circle}[[0, 0], 1] \right] \\
& \text{]]} \\
\text{(Mac) Out[12]} &= \\
\end{align*}
Kernels

Dirichlet kernel

(Mac) In[29]:=

\[ D_n[\lambda_] := \frac{\sin(\pi \lambda n)}{n \sin(\pi \lambda)} \]

With[\{n = 100\},
   Plot[D_n[\lambda], \{\lambda, -.1, .1\}, PlotRange -> All]
]


Fejer kernel

(Mac) In[30]:=

F_n[\lambda_] = n D_n[\lambda]^2;

For this comparison, we have to rescale the Fejer kernel.

(Mac) In[31]:=

With[\{n = 100\},
   Plot[{D_n[\lambda], F_n[\lambda]/n}, \{\lambda, -.1, .1\}, PlotRange -> All]
]

(Mac) Out[31]=

Example of leakage
In[18]:= 
\[
\text{zr} = 0.81 + 0.6 \; \text{Abs}[\text{zr}] \; \text{Arg}[\text{zr}] \; (2 \pi)
\]

\[
\phi_4[z_] := \phi[z] \left( 1 - \frac{z}{\text{zr}} \right) \left( 1 - \frac{z}{\text{Conjugate}[\text{zr}]} \right)
\]

Out[18]= \{1.00802, 0.101469\}

In[20]:= 
\[
f[\lambda_] = \text{Chop}\left[ \text{FullSimplify}\left[ \frac{1}{\phi_4[e^{2 \pi i \lambda}] \phi_4[e^{-2 \pi i \lambda}]} \right] \right]
\]

Out[20]= 
\[
0.0176687 / (0.602718 - 1. \cos[2 \pi \lambda] + 0.559736 \cos[4 \pi \lambda] - 0.194358 \cos[6 \pi \lambda] + 0.0330387 \cos[8 \pi \lambda])
\]

In[21]:= 
LogPlot[f[\lambda], \{\lambda, 0, 1/2\}]

Out[21]=

In[22]:= 
LogPlot[f[\lambda], \{\lambda, 0.09, 0.15\}]

Out[22]=

In[23]:=

\[\text{ContourPlot}[\text{Log}\left[\text{Abs}\left[\frac{1}{\phi_4(x+iy)\phi_4(x-iy)}\right]\right],\]
{x, -1.25, 1.25}, {y, -1.25, 1.25},
\text{Epilog} \rightarrow \text{Circle}[[0, 0], 1]\]

Out[23]=

Notice the impact of the sidelobes.

With[
{n = 100, \[xi] = 0.11},
\text{LogPlot}[\text{D}_n[\xi - \lambda f[\lambda], \{\lambda, 0, 0.5\}]\]
]

Out[33]=

\[\text{Plot}\]

Plot[f, \{x, x_{\text{min}}, x_{\text{max}}\}] \text{ generates \ a \ plot \ of \ } f \text{ \ as \ a \ function \ of \ } x \text{ \ from \ } x_{\text{min}} \text{ \ to \ } x_{\text{max}}. 

Plot[[f_1, f_2, \ldots], \{x, x_{\text{min}}, x_{\text{max}}\}] \text{ plots several functions } f_i.
In[34]:=
With[
{n = 100, \(\xi = 0.11\)},
LogPlot[F_{n}[\(\lambda - \xi\)] \(\lambda\), \{\(\lambda\), 0, 0.5\}, PlotRange\[\to\] All, PlotPoints\[\to\] 200]
]

In[34]=

Out[34]=

\begin{figure}
\centering
\includegraphics[width=\textwidth]{17_dft.nb}
\end{figure}