Spectral Representation

Overview

1. Motivation
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4. Spectral representation
5. Spectrum of ARMA processes

Caution: I adapted some of these notes from prior material that used so-called angular frequencies. Consequently, there's issues with misplaced 2π's here and there!

Motivation

Wold representation

$$X_t = \sum_{j=0}^{\infty} \underbrace{\psi_j}_{\text{constants}} \; \underbrace{W_{t-j}}_{\text{ortho r.v.}} \quad + \quad V_t$$

The Wold representation offers a very time-localized characterization of the variation in X_t, distributing the variation according to the weights ψ_j^2.

Alternative representation

$$X_t = \sum_j \underbrace{A_j}_{\text{ortho r.v.}} \; \underbrace{g_j(t)}_{\text{known func}}$$

This representation (ultimately with an integral rather than a sum) describes the variation as a superposition of functions. The functions are "global" in time, rather than isolated in time as in the Wold representation. You have to wonder how you can get something that is non-deterministic.

Candidate representation Expand the random phase model by adding other frequencies λ_j,

$$X_t = \sum_j A_j \cos(2\pi\lambda_j t) + B_j \sin(2\pi\lambda_j t)$$

where A_j and B_j are uncorrelated, mean-zero random variables with variance $\mathrm{Var}(A_j) = \mathrm{Var}(B_j) = \sigma_j^2$. Because time is discrete, we need only consider frequencies $-\frac{1}{2} < \lambda_j \le \frac{1}{2}$.

The calculations are simplified (ultimately) if we switch to the complex representation of the trig functions and write

$$X_t = \sum_j C_j e^{2\pi\imath\,\lambda_j t}, \quad -\tfrac{1}{2} < \lambda_{-j} = -\lambda_j \le \tfrac{1}{2}, \tag{1}$$

where C_j are complex-valued random variables with mean 0, uncorrelated, and have variance σ_j^2. For X_t to be real-valued, we add a bit of symmetry by having pairs of frequencies $\lambda_j, -\lambda_j$ and setting $C_{-j} = \overline{C_j}$.

Complex random variables Expected value works as usual, but conjugation appears in the covariance:

$$\mathrm{Cov}(X, Y) = \mathbb{E}\,(X - \mathbb{E}\,X)\overline{(Y - \mathbb{E}\,Y)}$$

The previous representation requires orthogonal coefficients, but we also have $C_{-j} = \overline{C_j}$. How's this possible? Write the random variable $C_j = A_j + \imath B_j$ where A_j and B_j real-valued r.v.s. Then orthogonality requires that

$$\begin{aligned} 0 = \mathrm{Cov}(C_j, C_{-j}) &= \mathbb{E}\,(A_j + \imath B_j)\overline{(A_j - \imath B_j)} = \mathbb{E}\,(A_j + \imath B_j)(A_j + \imath B_j) \\ &= \mathbb{E}\,(A_j^2 - B_j^2) + 2\imath\mathbb{E}\,(A_j B_j) \end{aligned}$$

Hence the underlying real-valued components must have equal variance and be uncorrelated, as in the definition of the random phase model.

Covariances The covariances of the process defined in (1) are (with the mean at zero)

$$\gamma(h) = \mathrm{Cov}(X_t, X_{t-h}) = \mathbb{E}\,(X_t \overline{X_{t-h}})$$

$$
\begin{aligned}
&= \sum_{j,k} \mathbb{E}\,(C_j \overline{C_k}) e^{2\pi\imath\,(\lambda_j t - \lambda_k (t-h))} \\
&= \sum_j \mathbb{E}\,|C_j|^2 e^{2\pi\imath\,\lambda_j h}
\end{aligned}
$$

The absence of t shows that the process is second-order stationary. Now imagine adding more frequencies to the sum. Define the function G with jumps of size $\mathbb{E}\,|C_j|^2/2$ at frequency λ_j and $-\lambda_j$. Then in the limit we can write (see eqn (C.1) in Appendix C)

$$
\gamma(h) = \int_{-1/2}^{1/2} e^{2\pi\imath\,h\lambda} dG(\lambda) \overset{\text{diff}}{=} \int_{-1/2}^{1/2} e^{2\pi\imath\,h\lambda} g(\lambda) d(\lambda) \ ,
$$

where the second form requires existence of a derivative $dG(\lambda)/d\lambda = g(\lambda)$. In fact, the first form of the Fourier representation holds for all stationary processes (shown later as Herglotz theorem), as does the associated "spectral" representation (1).

The Spectral Representation

Random measure The idea is to define a set function that associates random variable with sets. The set function in effect associates variation with frequency intervals. Write the random phase sum (1) as

$$
X_t = \sum_{j=-n}^{n} C_j e^{2\pi\imath\,t\lambda_j} = \int_{-1/2}^{1/2} e^{2\pi\imath\,t\lambda} Z_n(d\lambda) \tag{2}
$$

The frequencies are symmetrically laid out $\lambda_{-j} = -\lambda_j$ and $C_{-j} = \overline{C}_j$, and the C_j are uncorrelated with variance $\mathrm{Var}(C_j) = \sigma_j^2$. Z_n is the *random measure* defined by

$$
Z_n(S) = \sum_{\lambda_j \in S} C_j,
$$

for any set S. If we define

$$
F_n(S) = \sum_{\lambda_j \in S} \mathrm{Var}(C_j) \ ,
$$

then for sets A, B, we have $\mathbb{E}\,Z_n(A) = 0$ and

$$
\mathrm{Var}(Z_n(A)) = F_n(A), \quad \mathrm{Cov}(Z_n(A), Z_n(B)) = F_n(A \cap B).
$$

Also, by construction in the real-valued case, the pairing of frequencies and coefficients implies that

$$Z(-S) = \overline{Z(S)}\ .$$

In summary,

$$X_t = \int_{-1/2}^{1/2} e^{2\pi\imath\, t\lambda} Z_n(d\lambda) \quad \text{and} \quad \gamma(h) = \int_{-1/2}^{1/2} e^{2\pi\imath\, h\lambda} F_n(d\lambda)$$

Extension Consider the effect of adding more and more frequencies λ_j to the interval $(0, \pi)$. Since the trigonometric polynomials are dense in the collection of continuous functions on $[-\frac{1}{2}, \frac{1}{2}]$, we can approximate a continuous spectral distribution using a limiting sum of random phase models.

Spectral representation of a zero-mean, stationary process $\{X_t\}$ is

$$X_t = \int_{-1/2}^{1/2} e^{2\pi\imath\, t\lambda} Z(d\lambda)\ . \tag{3}$$

The *complex-valued random measure* Z is uncorrelated over disjoint intervals and has mean zero,

$$\begin{aligned} \mathrm{Cov}(Z([a,b]), Z([c,d])) &= \mathbb{E}\left[Z([a,b])\overline{Z([c,d])}\right] \\ &= 0 \quad \text{if} \quad [a,b] \cap [c,d] = \emptyset\ . \end{aligned}$$

The integral in (3) is over the half-open interval $(-\frac{1}{2}, \frac{1}{2}]$. In general, you will be little hurt by thinking of this as the full interval $[-\frac{1}{2}, \frac{1}{2}]$.

Link to covariances The variance of the random measure is linked to the covariance function γ of the process. Being careful to insert the conjugate, the spectral representation (3) makes this connection evident:

$$\begin{aligned} \gamma(h) = \mathbb{E}\ X_{t+h}\,\overline{X_t} &= \mathbb{E}\left(\int_{-1/2}^{1/2} e^{2\pi\imath\,(t+h)\lambda} Z(d\lambda) \int_{-1/2}^{1/2} e^{-2\pi\imath\, t\omega}\overline{Z(d\omega)}\right) \\ &= \int_{-1/2}^{1/2} e^{2\pi\imath\,(t+h)\lambda - 2\pi i t\omega} E[Z(d\lambda)\overline{Z(d\omega)}]d\lambda d\omega \\ &= \int e^{2\pi\imath\, h\lambda}\,\mathrm{Var}(Z(d\lambda))d\lambda \\ &= \int e^{2\pi\imath\, h\lambda} dF(d\lambda) \end{aligned}$$

The (non-stochastic) measure F is the spectral distribution function.

Applications

Uses Just as we can replace $\gamma(h)$ by its Fourier transform, we can replace X_t by its spectral representation. This substitution often leads to different insights into time series.

Isometry From the Hilbert space point-of-view, these manipulations produce an isometric isomorphism between second-order stationary processes and squared integrable functions $L^2(-\frac{1}{2}, \frac{1}{2}]$. The correspondence is

$$X_t \quad \Leftrightarrow \quad e^{2\pi\imath\, t\lambda}$$

(between a correlated sequence of r.v.'s and a collection of functions) with the inner products

$$\begin{aligned} \gamma(t-s) = \text{Cov}(X_t, X_s) &= \langle X_t,\, X_s \rangle \\ &= \langle e^{2\pi\imath\, t\lambda},\, e^{2\pi\imath\, s\lambda} \rangle_F = \int_{-1/2}^{1/2} e^{2\pi\imath\,(t-s)\lambda} dF(\lambda) \end{aligned}$$

Key theorem Let g and h denote integrable functions such that

$$\int |g(\lambda)|^2 dF(\lambda) < \infty, \quad \int |f(\lambda)|^2 dF(\lambda) < \infty\ .$$

Then the random variables

$$G = \int g(\lambda) Z(d\lambda), \quad H = \int h(\lambda) Z(d\lambda)$$

have finite variance and covariance

$$\text{Cov}(G,H) = \int g(\lambda)\overline{h(\lambda)} F(d\lambda) \quad \text{Var}(G) = \int |g(\lambda)|^2 F(d\lambda)\ .$$

Derivative of stochastic process More relevant in the case of continuous time where the integrals run over the real line and not from $-\frac{1}{2}$ to $\frac{1}{2}$. Assuming that the spectrum decreases rapidly as λ grows

$$\int_{-\infty}^{\infty} \lambda^2 F(d\lambda) < \infty\ ,$$

then the "derivative" of a process is

$$\frac{dX_t}{dt} = \int_{-\infty}^{\infty} i\lambda e^{it\lambda} Z(d\lambda)$$

with covariance function is

$$\gamma(h) = \int_{-\infty}^{\infty} \lambda^2 e^{ih\lambda} F(d\lambda).$$

Spectra of ARMA Processes

Linear filter The sum $Y_t = \sum_j a_j X_{t-j}$ is known as a time-invariant, linear filter. If the X_t are in a Hilbert space, then the sum is well-defined so long as $\sum_j a_j^2 < \infty$. If $a_j = 0,\ j < 0$, the filter is one-sided or causal. Causal ARMA processes have the infinite moving average form $X_t = \sum_{j=0}^{\infty} \psi_j w_{t-j}$ and are thus the output of a linear filtering of white noise.

Spectrum of linear filter Let $Y_t = \sum_j a_j X_{t-j}$ where $\{X_t\}$ is a stationary process with spectrum F_X and $\sum_j a_j^2 < \infty$, then ($a_j \in \mathbb{R}$)

$$\begin{aligned}
\gamma_Y(h) &= \langle \textstyle\sum_j a_j X_{t+h-j},\ \sum_k a_k X_{t-k} \rangle \\
&= \sum_{j,k} a_j a_k \langle X_{t+h-j},\ X_{t-k} \rangle \\
&= \sum_{j,k} a_j a_k \langle e^{2\pi\imath\,(t+h-j)\lambda},\ e^{2\pi\imath\,(t-k)\lambda} \rangle_{F_x} \\
&= \int_{-\pi}^{\pi} \sum_{j,k} a_j a_k e^{2\pi\imath\,(h-j+k)\lambda} dF_X(\lambda) \\
&= \int_{-\pi}^{\pi} e^{2\pi\imath\,h\lambda} |A(\lambda)|^2 dF_X(\lambda)\ ,
\end{aligned}$$

where the *transfer function* of the filter is $A(e^{2\pi\imath\,\lambda}) = \sum_j a_j e^{2\pi\imath\,j\lambda}$. Thus, the spectrum of $\{Y_t\}$ is

$$F_Y(\lambda) = \int_{(-1/2,\lambda]} |A(e^{2\pi\imath\,\lambda})|^2 dF_X(\lambda)\ . \tag{4}$$

At each frequency, the transfer function has a magnitude and phase. The filter magnifies the presence of some frequencies and attenuates others. It may also shift some frequencies more in time than others. It does not, however, mix frequencies. (Evil things happen in audio processing when nonlinearities in an amplifier mix frequencies.)

Using the spectral representation directly gives a different derivation. If $Y_t = \sum_j \theta_j X_{t-j}$, then

$$Y_t = \int \sum_j a_j e^{2\pi\imath (t-j)\lambda} Z_X(d\lambda) = \int e^{2\pi\imath t\lambda} A(e^{2\pi\imath\lambda}) Z_X(d\lambda) \ .$$

Hence,

$$\gamma_Y(h) = \int_{-1/2}^{1/2} |A(e^{2\pi\imath\lambda})|^2 F_X(d\lambda) \ .$$

Spectrum of white noise If $w_t \sim WN(0, \sigma^2)$, then the spectral density is constant, a mix of equal variance across all frequencies.

$$f(\lambda) = \sum_h \gamma(h) e^{-2\pi\imath\lambda h} = \sigma^2 \ .$$

Hence the name *white noise.*

Spectrum of AR If

$$\sum_j \phi_j X_{t-j} = w_t, \quad w_t \sim WN(0, \sigma^2),$$

then via the spectral representation, for all t,

$$\int_{-1/2}^{1/2} e^{2\pi\imath t\lambda} \sum_j \phi_j e^{-2\pi\imath j\lambda} dZ_X(\lambda) = \int_{-1/2}^{1/2} e^{2\pi\imath t\lambda} dZ_w(\lambda) \ .$$

Both sides have the same spectral density. Since $\phi(z)$ has no zeros on the unit circle, $\phi(\lambda) = \sum_j \phi_j e^{-2\pi\imath j\lambda}) \neq 0$ and we have

$$|\phi(\lambda)|^2 f_X(\lambda) = f_w(\lambda) = \sigma^2 \ ,$$

or

$$f_X(\lambda) = \frac{\sigma^2}{|\phi(\lambda)|^2} \ .$$

Spectra of ARMA If $\phi(B) Y_t = \theta(B) w_t$, then from (4) and existence of a spectral density f,

$$f_Y(\lambda) = |\psi(e^{2\pi\imath\lambda})|^2 f_w(\lambda) = \sigma^2 \frac{\theta(e^{2\pi\imath\lambda})\theta(e^{-2\pi\imath\lambda})}{\phi(e^{2\pi\imath\lambda})\phi(e^{-2\pi\imath\lambda})} \ .$$

Note the presence of the covariance generating function, so that all is consistent with earlier results.

Examples generated via R.

Herglotz Theorem and the Fourier Transform

Complex r.v. and covariance The use of complex-valued random variables complicates the definition of the covariance function.

- Dominated: $|\gamma(h)| \le \gamma(0) \quad \forall h$.
- Conjugate: $\gamma(h) = \text{Cov}(X_{t+h}, X_t) = \mathbb{E}\,(X_t - \mu)\overline{(X_s - \mu)}$.
- Hermitian: $\gamma(h) = \overline{\gamma(-h)}$.

As with real-valued processes, an Hermitian function K is the autocovariance function of a complex stationary process *iff* it is non-negative definite (n.n.d.) as generalized to complex processes:

$$\sum_{j,k=1}^{n} a_j \overline{a}_k K(j-k) \ge 0 \;, \tag{5}$$

for all $n > 0$ and n-tuples $a \in \mathbb{C}^n$.

Herglotz's Theorem (Appendix C) A complex-valued sequence $\{\gamma(h)\}$ is n.n.d. Hermitian *iff*

$$\gamma(h) = \int_{-1/2}^{1/2} e^{2\pi\imath\, h\lambda} dF(\lambda) \quad \forall h = 0, \pm 1, \ldots \;. \tag{6}$$

where the *spectral distribution* F is right-continuous, non-decreasing, and bounded on $(-\frac{1}{2}, \frac{1}{2}]$ and $F(-\frac{1}{2}) = 0$. If F is absolutely continuous with $F(\lambda) = \int_{-1/2}^{\lambda} f(\omega) d\omega$, then f is the spectral density function. (The continuous time version of this theorem is known as Bochner's theorem.)

Fourier transform pair From Herglotz's theorem, it follows that the covariances and spectrum are a Fourier transform pair, with the inverse transform being

$$F(\lambda) = \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} \frac{\gamma(j)}{ij} e^{-2\pi\imath\,\lambda j} \qquad \text{or} \qquad f(\lambda) = \sum_j \gamma(j) e^{-2\pi\imath\,\lambda j}$$

when the spectral density exists.

Analogy to CDF The spectral distribution F is essentially a rescaled cumulative distribution function. It does not integrate to 1, but rather integrates to the variance of the process since

$$\gamma(0) = \int_{-1/2}^{1/2} dF(\lambda) = F(\tfrac{1}{2}) - F(-\tfrac{1}{2}) = F(\tfrac{1}{2})$$

The covariance function thus resembles the characteristic function of a random variable. Both the spectral distribution and the CDF of a random variable are right-continuous, non-decreasing bounded functions, with at most countable jumps (think of mixtures of discrete and continuous random variables).

Anova The spectrum represents an *analysis of variance* of a stationary process. Rather than decompose variance by categories, the spectrum indicates how much of the variance of the process can be assigned to various frequency ranges. The total variance is $\gamma(0)$, whereas the variance associated with frequencies in the interval $[a, b]$ is $\int_a^b f(\lambda)d\lambda$.

Real-valued If X_t is real-valued, then $\gamma(h)$ is symmetric. In this case (assuming a spectral density), $f(\lambda)$ is symmetric as well,

$$f(\lambda) = \sum_j \gamma(j) e^{-2\pi\imath\lambda j} = f(-\lambda) \ ,$$

and

$$\gamma(h) = \int_{-1/2}^{1/2} \cos(2\pi h\lambda) f(\lambda) d\lambda \ .$$

In addition, the spectral density of a real-valued process has the following properties:

1. Non-negative
2. Integrable on $[-\frac{1}{2}, \frac{1}{2}]$

Proof of Herglotz's. $\Leftarrow$ Given the existence of the representation, write the Hermitian form and observe that γ must be Hermitian (the conjugate of an integral is the integral of the conjugate). The quadratic form is non-negative since (think of the integral as the expected value of a non-negative function)

$$\sum_{j,k} a_j \overline{a}_k \gamma(j-k) = \int_{-1/2}^{1/2} |\sum_j a_j e^{2\pi\imath j\lambda}|^2 dF(\lambda) \geq 0 \ .$$

⇒ The converse follows by construction. Start with an obvious guess, the partial sum $f_N(\lambda) = \sum_{r,s=1}^{n} e^{2\pi\imath\,\lambda r}\gamma(j)e^{-2\pi\imath\,\lambda s}/N$. (The eigenvalues of the covariance matrix are interesting at this point.) Since $\gamma(j)$ is n.n.d., a quadratic form assures us its positive and the result follows by a passage to the limit. The "trick" is to use the Toeplitz structure to reduce the quadratic form to a single sum,

$$0 \le f_N(\lambda) = \frac{\sum_{r,s=1}^{N} e^{2\pi\imath\,(r-s)\lambda}\gamma(r-s)}{N} = \frac{\sum_{|m|<N}(N-|m|)e^{-2\pi\imath\,m\lambda}\gamma(m)}{N}.$$

Define the absolutely continuous distribution $F_N(\lambda) = \int_{(-\frac{1}{2},\lambda]} f_N(\omega)d\omega$. Note that F_N has the needed properties, such as monotonicity. Using F_N in place of F gives

$$\begin{aligned}
\int_{-1/2}^{1/2} e^{2\pi\imath\,h\lambda} dF_N(\lambda) &= \int f_N(\lambda)e^{2\pi\imath\,h\lambda}d\lambda \\
&= \sum_m (1-|h|/N)\gamma(h)\int e^{2\pi\imath\,(h-m)\lambda}d\lambda/(2\pi) \\
&= (1-|h|/N)\gamma(h), \quad |h|<N.
\end{aligned}$$

and is zero otherwise. It now follows from the theory of weak convergence or the Helly selection theorem that since $F_N(\pi) = \gamma(0)$ is bounded $\forall N$, we can choose a subsequence F_{N_k} such that (6) holds in the limit.

Absolute summability of the covariance function implies that the spectral density function exists. From the proof of Herglotz's theorem,

$$0 \le f_N(\lambda) = \sum_{|h|<N}(1-|h|/N)e^{ih\lambda}\gamma(h).$$

This sum is dominated by $\sum|\gamma(h)|$ and so dominated convergence allows us to interchange limits.

Appendix: Orthogonal increments processes

Orthogonal increments process Defined as a continuous time process $Z(\lambda), -\pi \le \lambda \le \pi$, for which

1. Mean zero: $E\ Z(\lambda) = 0$, or using inner products $\langle Z(\lambda), 1\rangle = 0$.

2. Finite variance: $\text{Var } Z(\lambda) = \langle Z(\lambda),\, Z(\lambda)\rangle\ < \infty$.
3. Orthogonal: $\text{Cov}(Z(\lambda_4)-Z(\lambda_3), Z(\lambda_2)-Z(\lambda_1)) = \langle Z(\lambda_4) - Z(\lambda_3),\, Z(\lambda_2) - Z(\lambda_1))\rangle = 0$ if $(\lambda_1, \lambda_2] \cap (\lambda_3, \lambda_4] = \emptyset$.
4. Right continuous: $\lim_{\delta\downarrow 0} \|Z(\lambda+\delta) - Z(\lambda)\| = 0$.

Note that the inner product requires conjugating the second term.

Examples of such processes are

1. Brownian motion $B(\lambda)$ on $[-1/2, 1/2]$, for which $F(\lambda) = \lambda + \frac{1}{2}$.
2. Starting from the Poisson process $N(\lambda)$ with arrival rate μ, let $Z(\lambda) = N(\lambda) - \mu(\lambda + \frac{1}{2})$.

In both cases, the variance of the process grows as a linear function of λ.

Distribution function. Define the function

$$F(\lambda) = \|Z(\lambda) - Z(-1/2)\|^2 .$$

From the right-continuity of Z, it follows that F is right-continuous as well:

$$F(-1/2) = 0 \quad \text{and} \quad \lim_{\delta\downarrow 0} F(\lambda+\delta) = F(\lambda)\ .$$

From the orthogonality of Z, F is monotone:

$$F(\lambda) = \|Z(\lambda) - Z(\omega) + Z(\omega) - Z(-\pi)\|^2 \geq F(\omega), \quad \lambda > \omega .$$

Hence F behaves like a *multiple of the CDF* of a random variable defined on $[-1/2, 1/2]$. However, F only characterizes second-order moments, *not* probabilities. For both Brownian motion and the normalized Poisson processes, F is linear in λ.

Appendix: Stochastic integrals

Define a stochastic integral by starting with step functions, then extend to other functions as in Lebesgue integration. (This is *not* the stochastic integral as defined by Ito.)

Stochastic integral for step functions Define the stochastic integral of a step function

$$f(\lambda) = \sum_{j=0}^{n} f_j I_{(\lambda_j, \lambda_{j+1}]}(\lambda)$$

for the partition $-1/2 = \lambda_0 < \lambda_2 < \cdots < \lambda_{n+1} = 1/2$ as

$$I(f) := \int_{(-\pi,\pi]} f(\lambda) Z(d\lambda) := \sum_{j=0}^{n} f_j \left(Z(\lambda_{j+1}) - Z(\lambda_j) \right) \tag{7}$$

Properties of this integral that maps a step function to a random variable include:

1. The resulting random variables have finite variance if the step function is $L^2[-1/2, 1/2]$ w.r.t. F:

$$\mathrm{Var}(I(f)) = \sum_j f_j^2 \left(F(\lambda_{j+1}) - F(\lambda_j) \right)$$

2. We obtain an inner product for the resulting r.v.'s from the definition. Assuming both use a common partition,

$$\begin{aligned} \langle I(f), I(g) \rangle &= \mathrm{Cov}(I(f), I(g)) \\ &= \int_{-1/2}^{1/2} f \overline{g} dF(\lambda) \\ &= \sum_j f_j \overline{g}_j \left(F(\lambda_{j+1}) - F(\lambda_j) \right) \end{aligned}$$

3. It's linear, and from the first property, bounded.

BLT theorem. Let T be a linear transformation from a normed linear space V_1 into a complete normed space V_2. Then T can be uniquely extended to a bounded linear transformation $\tilde{T}$ (with the same bound) from the completion of V_1 into V_2.

The theorem gives a means for attacking hard problems. First define a linear operator on a dense subset. Then use the BLT theorem to show that the operator may be extended to the whole space. It is often useful to note the following equivalence: If T is a linear transformation between normed linear spaces, the following are equivalent:

1. T is continuous at a point (in particular, zero).
2. T is continuous everywhere.
3. T is bounded.

Extension. Our definition of the stochastic integral $I(f)$ maps the $L^2(dF)$ integrable step functions to L^2 random variables. The step functions are dense in L^2, and both spaces meet the conditions of the BLT theorem. Consequently, there is an extension of $I(f)$ to all of $L^2(dF)$. Also, the results above imply that the integral $I(f)$ is continuous in f.

Spectral representation of a process is thus defined (noting that the functions $g(x) = e^{2\pi\imath tx}$ are integrable w.r.t. F) as

$$X_t = I(g) = \int_{-1/2}^{1/2} e^{2\pi\imath t\lambda} dZ(\lambda) \; .$$