Neyman-Scott, the Square Array and efficient estimation.

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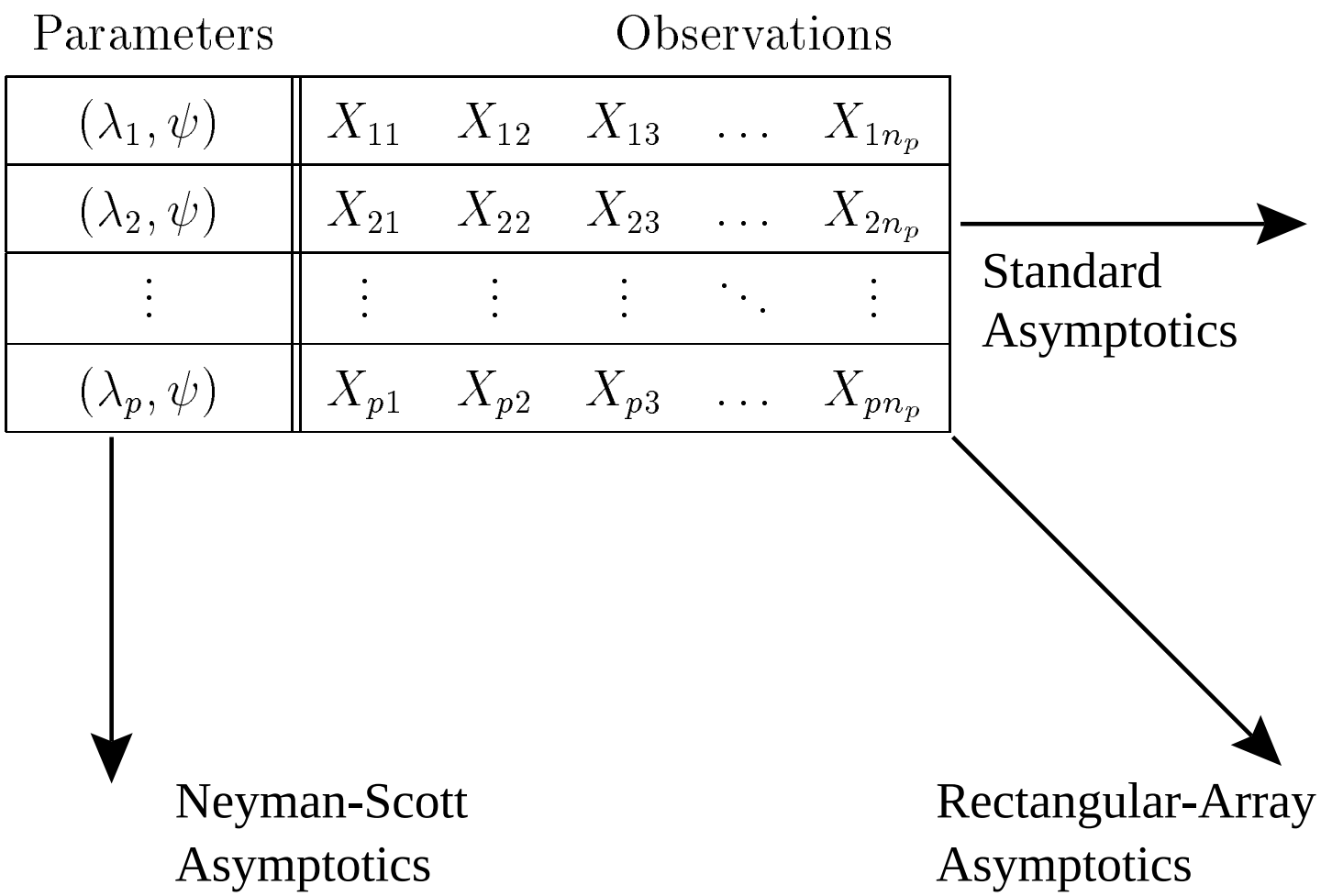
Three Embeddings

Number of strata $= p$.
Stratum density $f_i = f(x_{ij}; \psi, \lambda_i)$.

- Independent identically distributed within strata
- Independent between strata



Why?

- Theoretical: insight
- Practical: more appoximation theorems
- Efficiency result: a baseline

Other work

- Andersen
- Mallows
- Haberman
- Tak
- Portnoy
- Pfanzagl

Example

Oneway ANOVA.

Parameter of interest is σ^2.

Nuisance parameter is μ_i, taken as FIXED.

The stratum density is $f(x_{ij}; \sigma^2, \mu_i)$, $i = 1, \ldots, p$, $j = 1, \ldots, n_p$.

Maximum likelihood limiting distributions

$$\begin{aligned}
\text{Standard:} \quad & \hat{\sigma}^2 - \sigma^2 \xrightarrow{p} 0 \\
\text{Neyman-S:} \quad & \hat{\sigma}^2 - \sigma^2 \xrightarrow{p} \{-1/n\}\sigma^2 \\
\text{SA, } n = p\text{:} \quad & \hat{\sigma}^2 - \sigma^2 \xrightarrow{p} 0
\end{aligned}$$

$$\begin{aligned}
\text{Standard:} \quad & \sqrt{n\,p}\left(\hat{\sigma}^2 - \sigma^2\right) \xrightarrow{\ell} N(0, 2\sigma^4) \\
\text{SA, } n = p\text{:} \quad & \sqrt{n\,p}\left(\hat{\sigma}^2 - \sigma^2\right) \xrightarrow{\ell} N(-\sigma^2, 2\sigma^4)
\end{aligned}$$

Table 1: Properties of the *mle*, $\hat{\psi}$, in the three asymptotic embeddings.

Asymptotics	Consistency	Efficiency
Standard	✓	✓
Square Array	✓	✗
Neyman-Scott	✗	✗

Asymptotic Efficiency in the presence of nuisance parameters.

Standard:

$$\sqrt{n}\,(\psi_n - \psi) \xrightarrow{\ell} N(0, c^{-1}),$$

where $c = (i^{\psi\psi})^{-1} = i_{\psi\psi} - i_{\psi\lambda} i_{\lambda\lambda}^{-1} i_{\lambda\psi}$.

$$M_1 = E\left\{ \begin{pmatrix} U_0 \\ V_1 \end{pmatrix} (U_0, V_1) \right\} = \begin{pmatrix} i_{\psi\psi} & i_{\psi\lambda} \\ i_{\lambda\psi} & i_{\lambda\lambda} \end{pmatrix}.$$

Infinite nuisance parameter

$$\sqrt{n\,p}\,(\psi_n - \psi) \xrightarrow{\ell} N(0, \overline{c}^{-1}),$$

where $\overline{c} = \lim_{p\to\infty} \frac{1}{p} \sum_{k=1}^{p} (i_k^{\psi\psi})^{-1}$ and $(i_k^{\psi\psi})^{-1}$ is the information in the k^{th} stratum.

Anderson 1970. Suggestion for NS asymptotics, but not attainable in general.

Is it a sensible here in the Square Array?

Since the λ_i is an infinite sequence we need some structure if we are going to say something useful.

- Bounded set.
- Lindenberg-Feller type conditions.
- Come from a distribution.

Bias removal

MLE comes from score equations. Score equations are mean zero but substitution versions are NOT.

Example: In the i^{th} stratum .

$$U_0 = \frac{\partial}{\partial \psi} ln\, f(\psi, \lambda_i, \mathbf{x_i}) = -\frac{n}{2\,\psi} + \frac{1}{2\,\psi^2}\sum_{j=1}^{n}(x_{ij} - \lambda_i)^2 \quad (1)$$

$$V_1 = \frac{\partial}{\partial \lambda_i} ln\, f(\psi, \lambda_i, \mathbf{x_i}) = \frac{1}{\psi}\sum_{j=1}^{n}(x_{ij} - \lambda_i) \quad (2)$$

The solution to (2) is $\hat{\lambda}_i = \overline{\mathbf{x}}_i$. If this estimate for λ_i is "plugged" back into (1), then

$$E\left[-\frac{n}{2\,\psi} + \frac{1}{2\,\psi^2}\sum_{j=1}^{n}(x_{ij} - \hat{\lambda}_i)^2\right] = -\frac{1}{2\,\psi}. \quad (3)$$

The estimator $\hat{\psi}$ is obtained from solving

$$-\frac{n}{2\,\psi} + \frac{1}{2\,\psi^2}\sum_{j=1}^{n}(x_{ij} - \hat{\lambda}_i)^2 = 0,$$

but (3) shows that this doesn't even equal 0 on average! The consequence is an inconsistent estimator of ψ.

Fix this up.

Good estimating equations

Some good score functions/ estimating equations:

If we are lucky can condition on a complete and sufficient statistic for the nuisance parameters:

$$\text{Conditional Score} \qquad U_c = U_0 - E(U_0 \,|\, T).$$

Optimality results from Godambe (1976) and Lindsay (1982).

More generally U_∞.

$$U_\infty = U_0 - \Pi_\infty U_0.$$

$$E_\lambda\{U_\infty(\lambda_0)\} = 0 \quad \text{for all} \quad \lambda, \lambda_0, \psi_0.$$

Π_∞ is projection onto

$$\left\{ \frac{f(Y;\psi_0,\lambda)}{f(Y;\psi_0,\lambda_0)} - 1 \right\}, \lambda \in \Lambda.$$

Space generated by the centered likelihood ratios, Λ_∞. (Barankin(1949)). Characterization: the space of Locally Minimum Variance Unbiased Estimators of functions of λ for ψ_0 fixed.

A basis for Λ_∞

Basis for this space is $\{V_i, V_{ij}, \ldots\}$, where

$$V_{i_1 \ldots i_t} = \frac{\partial^t f}{\partial \lambda_{i_1} \ldots \partial \lambda_{i_t}} \Big/ f.$$

Examples

$$V_i = \frac{\partial f}{\partial \lambda_i} \Big/ f \qquad V_{ij} = \frac{\partial^2 f}{\partial \lambda_i \partial \lambda_j} \Big/ f.$$

So use (univariate λ.)

$$\begin{aligned} U_1 &= U_0 - \rho_1 V_1 && \text{Neyman's } C(\alpha). \\ U_2 &= U_0 - \rho_{11} V_1 - \rho_{12} V_{11} && \text{Small and McLeish} \end{aligned}$$

Claim the root of U_2 is just right for obtaining full efficiency in the rectangular array.

Facts about the Bhattachrayya Basis

Recurrence relation:

$$V_{i_1 \ldots i_{t-1} i_t} = (\partial / \partial \lambda_t) V_{i_1 \ldots i_{t-1}} + V_{i_1 \ldots i_{t-1}} V_{i_t}.$$

Example

$$V_{ij} = (\partial / \partial \lambda_j) V_i + V_i V_j.$$

Expectation Identity: If $T(\lambda) \perp \{V_{i_1}, V_{i_1 i_2}, \ldots V_{i_1 \ldots i_{t-1}}\}$, υ_1 and υ_2 are partitions of the indices $\{i_1, \ldots, i_t\}$ and q is the number of indices in υ_1 then

$$E(T_{\upsilon_1} V_{\upsilon_2}) = (-1)^{t-q} E(T_{i_1 \ldots i_t}) \qquad 0 \leq q \leq t.$$

Example

$$E(U_1'') = -E(U_1' V_1) = E(U_1 V_{11}).$$

Idea of proof of efficiency of U_2 root.

Call the root $\tilde{\psi}$.

Want expansion of

$$\sqrt{n\,p}\left(\tilde{\psi}-\psi\right).$$

Technique: von-Mises expansion. For the i^{th} stratum define

$$F_i(\epsilon_i) = F_{i0} + \epsilon_i\sqrt{n_i}(\hat{F}_i - F_{i0}),$$

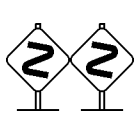
so that the derivative with respect to ϵ_i is

$$\sqrt{n_i}(\hat{F}_i - F_i) = \Delta_{in}.$$

The second order von-Mises expansion is defined by

$$\psi\left(\frac{1}{\sqrt{n_1}},\ldots,\frac{1}{\sqrt{n_p}}\right) - \psi(0,\ldots,0) = \sum_k \psi'_k\Big|_{\epsilon_k=0}\frac{1}{\sqrt{n_k}} +$$

$$\frac{1}{2}\sum_{kl}\psi''_{kl}\Big|_{\epsilon_k=\epsilon_l=0}\frac{1}{\sqrt{n_k}}\frac{1}{\sqrt{n_l}} + Rem(\hat{F}-F_0)$$



Solving for ψ'_k and ψ''_{kl}

Obtain by differentiating

$$n_i^{-1} U_{2i} = \int U_{1ij} dF(\epsilon_i) -$$

$$\rho_i(\psi, \lambda) \left[\left(\int V_{1ij} dF(\epsilon_i) \right)^2 + \epsilon_i^2 \left(\int V_{1ij}^{\lambda_i} dF(\epsilon_i) \right) + o(\epsilon_j^4) \right].$$

Working form (use $\hat{\lambda}_\psi$ as estimates for the nuisance parameters):

$$U_2(\boldsymbol{\epsilon}) = \sum_i \left\{ \int U_{1ij}(\psi(\boldsymbol{\epsilon}_1), \lambda_i(\psi(\boldsymbol{\epsilon}_1), \epsilon_{2i})) dF_{\epsilon_{3i}} - \right.$$

$$\underbrace{\left. \rho_i(\psi(\boldsymbol{\epsilon}_1), \lambda_i(\psi(\boldsymbol{\epsilon}_1), \epsilon_{2i})) \left(\epsilon_{4i}^2 \int V_{1ij}^{\lambda_i}(\psi(\boldsymbol{\epsilon}_1), \lambda_i(\psi(\boldsymbol{\epsilon}_1), \epsilon_{2i})) dF_{\epsilon_{3i}} \right) \right\}}_{\text{Correction term}}.$$

$$U_2(\boldsymbol{\epsilon}) = U_1(\boldsymbol{\epsilon}) + \text{Correction}.$$

First order terms

$$\sqrt{n\,p}\sum_k \psi_k'\Big|_{\epsilon_k=0}\frac{1}{\sqrt{n_k}} \xrightarrow{\ell} N(0,\overline{c}^{-1}).$$

Second order terms

$$\sqrt{n\,p}\,\frac{1}{2}\sum_{kl}\psi_{kl}''\Big|_{\epsilon_k=\epsilon_l=0}\frac{1}{\sqrt{n_k}}\frac{1}{\sqrt{n_l}}$$

has components ψ'' of the form

$$\sum_{kl}\psi_{kl}''O(p)+\sum_k\tilde{U}_k+O_p(\sqrt{p})=0,$$

where the $\tilde{U}_k$ are mean zero when U_2 is used but NOT mean zero if U_1 is used.

So

$$\sum_{kl}\psi_{kl}''=\frac{-\left(\sum_k\tilde{U}_k+O_p(\sqrt{p})\right)}{O(p)}\xrightarrow{p}\text{Const.}$$

The constant.

$$-\frac{1}{2}\frac{E(U_{1i}V_{2i}^*)}{\overline{c}E(V_{1i}^2)}.$$

$$M_2 = E\left\{\begin{pmatrix} U_0 \\ V_1 \\ V_2 \end{pmatrix}(U_0, V_1, V_2)\right\}.$$

Example: (for normal, many means).

$$\begin{pmatrix} \frac{1}{2\sigma^4} & 0 & \frac{1}{\sigma^4} \\ 0 & \frac{1}{\sigma^2} & 0 \\ \frac{1}{\sigma^4} & 0 & \frac{2}{\sigma^4} \end{pmatrix}.$$

$$E(U_{1i}V_{2i}^*) = \frac{1}{\sigma^4} \qquad \overline{c} = \frac{1}{2\sigma^4} \qquad E(V_{1i}^2) = \frac{1}{\sigma^2},$$

so

$$Const = \frac{-\frac{1}{\sigma^4}}{\frac{1}{2\sigma^4}\frac{1}{\sigma^2}} = -\sigma^2.$$

Simulations

Second order information correction:

True info

$$\begin{array}{rcl} InfoU_2(\hat{\lambda}) & = & InfoU_1 - \left(\boxed{\frac{1}{2}\kappa^2} + \boxed{\text{Term 2}} + \boxed{\text{Term 3}} \right). \\ & & O(n) \quad - \quad O(1) \end{array}$$

Oneway ANOVA

Four pivotals:

1.
$$\frac{\sqrt{n\,p}(\hat{\psi} - \psi)}{\sqrt{2}\psi}$$

2.
$$\frac{\sqrt{n\,p}(\hat{\psi} - \psi)}{\sqrt{2}\psi} - \frac{1}{\sqrt{2}}$$

3.
$$\frac{\sqrt{n\,p}(\tilde{\psi} - \psi)}{\sqrt{2}\psi}$$

4.
$$a(\kappa, i)\frac{\sqrt{n\,p}(\tilde{\psi} - \psi)}{\sqrt{2}\psi}$$

$$a(k, i) = 1 - \frac{\kappa^2}{4\,n\,i}$$

Exact 95% coverage rates

$n = p$	1	2	3	4
2	0.9915	0.9770	0.9080	0.9420
3	0.9866	0.9737	0.9254	0.9511
4	0.9533	0.9730	0.9271	0.9547
5	0.9355	0.9716	0.9298	0.9538
10	0.9089	0.9620	0.9388	0.9510
15	0.9021	0.9580	0.9424	0.9504
20	0.8991	0.9559	0.9443	0.9502

Interpretation

$$\hat{\psi} = \frac{1}{p}\frac{1}{n}\sum (x_{ij} - \overline{x}_i)^2$$

$$\tilde{\psi} = \frac{1}{p}\frac{1}{n-1}\sum (x_{ij} - \overline{x}_i)^2$$

If use information correction then get

$$a(\kappa, i)\sqrt{np}\ (\tilde{\psi} - \psi) = \sqrt{(n-1)p}\ (\tilde{\psi} - \psi).$$

Circular Normal problem

(X,Y) come from a bivariate normal with mean lying on a circle with unknown radius ρ.

Simulated 95% coverage rates

n = p	1	2	3	4
2	95052	95831	93715	94483
3	94693	95420	94287	94801
4	94597	95326	94568	94945
5	94586	95281	94668	94988
10	94348	95119	94834	94986
20	94322	95142	95007	95074

Maximizing the log-likelihood

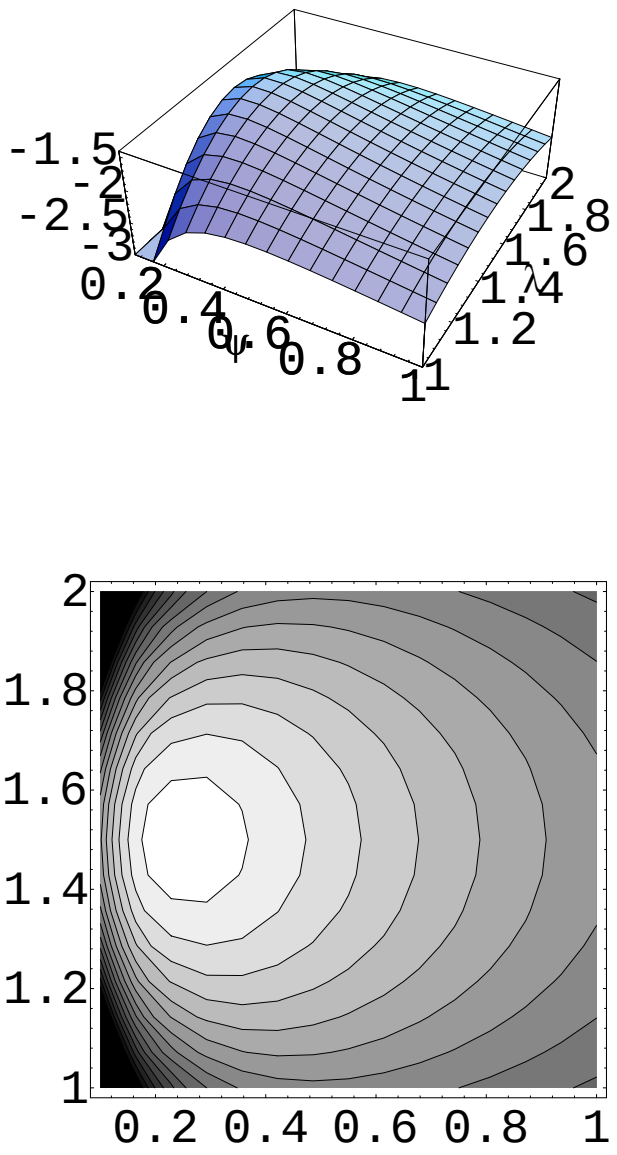


Figure 1: Normal variance and mean