Neyman-Scott Problems and Projected Scores.

RICHARD P. WATERMAN

Department of Statistics, University of Pennsylvania

BRUCE G. LINDSAY

Department of Statistics, Pennsylvania State University

Warsaw, November 1994

waterman@compstat.wharton.upenn.edu

http://compstat.wharton.upenn.edu:8001/~waterman/homepage.html

Three Embeddings

Number of strata $= p$.

Stratum density $f_i = f(x_{ij}; \psi, \lambda_i)$.

- Independent identically distributed within strata
- Independent between strata

Parameters	Observations				
(λ_1, ψ)	X_{11}	X_{12}	X_{13}	$\ldots$	X_{1n_p}
(λ_2, ψ)	X_{21}	X_{22}	X_{23}	$\ldots$	X_{2n_p}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
(λ_p, ψ)	X_{p1}	X_{p2}	X_{p3}	$\ldots$	X_{pn_p}

Standard Asymptotics

Neyman-Scott Asymptotics

Rectangular-Array Asymptotics

Why?

- Theoretical: insight
- Practical: more appoximation theorems
- Efficiency result: a baseline

Other work

- Andersen
- Mallows
- Haberman
- Tak
- Portnoy
- Pfanzagl

Example

Oneway ANOVA.

Parameter of interest is σ^2.

Nuisance parameter is μ_i, taken as FIXED.

The stratum density is $f(x_{ij}; \sigma^2, \mu_i)$, $i = 1, \ldots, p$, $j = 1, \ldots, n_p$.

Maximum likelihood limiting distributions

$$\begin{array}{rccl} \text{Standard:} & \hat{\sigma}^2 - \sigma^2 & \xrightarrow{p} & 0 \\ \text{Neyman-S:} & \hat{\sigma}^2 - \sigma^2 & \xrightarrow{p} & \{-1/n\}\sigma^2 \\ \text{SA, } n = p\text{:} & \hat{\sigma}^2 - \sigma^2 & \xrightarrow{p} & 0 \end{array}$$

$$\begin{array}{rccl} \text{Standard:} & \sqrt{n\,p}\left(\hat{\sigma}^2 - \sigma^2\right) & \xrightarrow{\ell} & N(0, 2\sigma^4) \\ \text{SA, } n = p\text{:} & \sqrt{n\,p}\left(\hat{\sigma}^2 - \sigma^2\right) & \xrightarrow{\ell} & N(-\sigma^2, 2\sigma^4) \end{array}$$

Table 1: Properties of the *mle*, $\hat{\psi}$, in the three asymptotic embeddings.

Asymptotics	Consistency	Efficiency
Standard	✓	✓
Square Array	✓	✗
Neyman-Scott	✗	✗

Asymptotic Efficiency in the presence of nuisance parameters.

Standard:

$$\sqrt{n}\,(\psi_n - \psi) \xrightarrow{\ell} N(0, c^{-1}),$$

where $c = (i^{\psi\psi})^{-1} = i_{\psi\psi} - i_{\psi\lambda} i_{\lambda\lambda}^{-1} i_{\lambda\psi}$.

$$M_1 = E\left\{ \begin{pmatrix} U_0 \\ V_1 \end{pmatrix} (U_0, V_1) \right\} = \begin{pmatrix} i_{\psi\psi} & i_{\psi\lambda} \\ i_{\lambda\psi} & i_{\lambda\lambda} \end{pmatrix}.$$

Infinite nuisance parameter

$$\sqrt{n\,p}\,(\psi_n - \psi) \xrightarrow{\ell} N(0, \overline{c}^{-1}),$$

where $\overline{c} = \lim_{p\to\infty} \frac{1}{p} \sum_{k=1}^{p} (i_k^{\psi\psi})^{-1}$ and $(i_k^{\psi\psi})^{-1}$ is the information in the k^{th} stratum.

Anderson 1970. Suggestion for NS asymptotics, but not attainable in general.

Is it a sensible here in the Square Array?

Since the λ_i is an infinite sequence we need some structure if we are going to say something useful.

- Bounded set.
- Lindenberg-Feller type conditions.
- Come from a distribution.

Bias removal

MLE comes from score equations. Score equations are mean zero but substitution versions are NOT.

Example: In the i^{th} stratum .

$$U_0 = \frac{\partial}{\partial \psi} ln\, f(\psi, \lambda_i, \mathbf{x_i}) = -\frac{n}{2\,\psi} + \frac{1}{2\,\psi^2} \sum_{j=1}^{n} (x_{ij} - \lambda_i)^2 \quad (1)$$

$$V_1 = \frac{\partial}{\partial \lambda_i} ln\, f(\psi, \lambda_i, \mathbf{x_i}) = \frac{1}{\psi} \sum_{j=1}^{n} (x_{ij} - \lambda_i) \quad (2)$$

The solution to (2) is $\hat{\lambda}_i = \overline{\mathbf{x}}_i$. If this estimate for λ_i is "plugged" back into (1), then

$$E\left[-\frac{n}{2\,\psi} + \frac{1}{2\,\psi^2} \sum_{j=1}^{n} (x_{ij} - \hat{\lambda}_i)^2 \right] = -\frac{1}{2\,\psi}. \quad (3)$$

The estimator $\hat{\psi}$ is obtained from solving

$$-\frac{n}{2\,\psi} + \frac{1}{2\,\psi^2} \sum_{j=1}^{n} (x_{ij} - \hat{\lambda}_i)^2 = 0,$$

but (3) shows that this doesn't even equal 0 on average! The consequence is an inconsistent estimator of ψ.

Fix this up.

Good estimating equations

Some good score functions/ estimating equations:

If we are lucky can condition on a complete and sufficient statistic for the nuisance parameters:

$$\text{Conditional Score} \qquad U_c = U_0 - E(U_0 \mid T).$$

Optimality results from Godambe (1976) and Lindsay (1982).

More generally U_∞.

$$U_\infty = U_0 - \Pi_\infty U_0.$$

$$E_\lambda\{U_\infty(\lambda_0)\} = 0 \quad \text{for all} \quad \lambda, \lambda_0, \psi_0.$$

Π_∞ is projection onto

$$\left\{ \frac{f(Y;\psi_0,\lambda)}{f(Y;\psi_0,\lambda_0)} - 1 \right\}, \lambda \in \Lambda.$$

Space generated by the centered likelihood ratios, Λ_∞. (Barankin(1949)). Characterization: the space of Locally Minimum Variance Unbiased Estimators of functions of λ for ψ_0 fixed.

A basis for Λ_∞

Basis for this space is $\{V_i, V_{ij}, \ldots\}$, where

$$V_{i_1 \ldots i_t} = \frac{\partial^t f}{\partial \lambda_{i_1} \ldots \partial \lambda_{i_t}} \Big/ f.$$

Examples

$$V_i = \frac{\partial f}{\partial \lambda_i} \Big/ f \qquad V_{ij} = \frac{\partial^2 f}{\partial \lambda_i \partial \lambda_j} \Big/ f.$$

So use (univariate λ.)

$$\begin{aligned} U_1 &= U_0 - \rho_1 V_1 && \text{Neyman's } C(\alpha). \\ U_2 &= U_0 - \rho_{11} V_1 - \rho_{12} V_{11} && \text{Small and McLeish} \end{aligned}$$

Claim the root of U_2 is just right for obtaining full efficiency in the rectangular array.

Facts about the Bhattachrayya Basis

Recurrence relation:

$$\boxed{V_{i_1\ldots i_{t-1}i_t} = (\partial/\partial\lambda_t)V_{i_1\ldots i_{t-1}} + V_{i_1\ldots i_{t-1}}V_{i_t}.}$$

Example

$$V_{ij} = (\partial/\partial\lambda_j)V_i + V_iV_j.$$

Expectation Identity: If $T(\lambda) \perp \{V_{i_1}, V_{i_1i_2}, \ldots V_{i_1\ldots i_{t-1}}\}$, υ_1 and υ_2 are partitions of the indices $\{i_1, \ldots, i_t\}$ and q is the number of indices in υ_1 then

$$\boxed{E(T_{\upsilon_1}V_{\upsilon_2}) = (-1)^{t-q}E(T_{i_1\ldots i_t}) \qquad 0 \le q \le t.}$$

Example

$$E(U_1'') = -E(U_1'V_1) = E(U_1V_{11}).$$

Idea of proof of efficiency of U_2 root.

Call the root $\tilde{\psi}$.

Want expansion of

$$\sqrt{n\,p}\left(\tilde{\psi} - \psi\right).$$

Technique: von-Mises expansion. For the i^{th} stratum define

$$F_i(\epsilon_i) = F_{i0} + \epsilon_i \sqrt{n_i}(\hat{F}_i - F_{i0}),$$

so that the derivative with respect to ϵ_i is

$$\sqrt{n_i}(\hat{F}_i - F_i) = \Delta_{in}.$$

The second order von-Mises expansion is defined by

$$\psi\left(\frac{1}{\sqrt{n_1}}, \ldots, \frac{1}{\sqrt{n_p}}\right) - \psi(0, \ldots, 0) = \sum_k \psi'_k\Big|_{\epsilon_k = 0} \frac{1}{\sqrt{n_k}} +$$

$$\frac{1}{2}\sum_{kl} \psi''_{kl}\Big|_{\epsilon_k = \epsilon_l = 0} \frac{1}{\sqrt{n_k}}\frac{1}{\sqrt{n_l}} + Rem(\hat{F} - F_0)$$

Solving for ψ'_k and ψ''_{kl}

Obtain by differentiating

$$n_i^{-1} U_{2i} = \int U_{1ij} dF(\epsilon_i) -$$

$$\rho_i(\psi, \lambda) \left[\left(\int V_{1ij} dF(\epsilon_i) \right)^2 + \epsilon_i^2 \left(\int V_{1ij}^{\lambda_i} dF(\epsilon_i) \right) + o(\epsilon_j^4) \right].$$

Working form (use $\hat{\lambda}_\psi$ as estimates for the nuisance parameters):

$$U_2(\boldsymbol{\epsilon}) = \sum_i \left\{ \int U_{1ij}(\psi(\boldsymbol{\epsilon}_1), \lambda_i(\psi(\boldsymbol{\epsilon}_1), \epsilon_{2i})) dF_{\epsilon_{3i}} - \right.$$

$$\left. \underbrace{\rho_i(\psi(\boldsymbol{\epsilon}_1), \lambda_i(\psi(\boldsymbol{\epsilon}_1), \epsilon_{2i})) \left(\epsilon_{4i}^2 \int V_{1ij}^{\lambda_i}(\psi(\boldsymbol{\epsilon}_1), \lambda_i(\psi(\boldsymbol{\epsilon}_1), \epsilon_{2i})) dF_{\epsilon_{3i}} \right) \right\}}_{\text{Correction term}}.$$

$$U_2(\boldsymbol{\epsilon}) = U_1(\boldsymbol{\epsilon}) + \text{Correction}.$$

First order terms

$$\sqrt{n\,p}\,\sum_k \psi'_k\bigg|_{\epsilon_k=0}\frac{1}{\sqrt{n_k}} \xrightarrow{\ell} N(0,\overline{c}^{-1}).$$

Second order terms

$$\sqrt{n\,p}\,\frac{1}{2}\sum_{kl}\psi''_{kl}\bigg|_{\epsilon_k=\epsilon_l=0}\frac{1}{\sqrt{n_k}}\frac{1}{\sqrt{n_l}}$$

has components ψ'' of the form

$$\sum_{kl}\psi''_{kl}O(p)+\sum_k \tilde{U}_k + O_p(\sqrt{p}) = 0,$$

where the $\tilde{U}_k$ are mean zero when U_2 is used but NOT mean zero if U_1 is used.

So

$$\sum_{kl}\psi''_{kl} = \frac{-\left(\sum_k \tilde{U}_k + O_p(\sqrt{p})\right)}{O(p)} \xrightarrow{p} \text{Const.}$$

The constant.

$$-\frac{1}{2}\frac{E(U_{1i}V_{2i}^*)}{\overline{c}E(V_{1i}^2)}.$$

$$M_2 = E\left\{\begin{pmatrix} U_0 \\ V_1 \\ V_2 \end{pmatrix}(U_0, V_1, V_2)\right\}.$$

Example: (for normal, many means).

$$\begin{pmatrix} \frac{1}{2\sigma^4} & 0 & \frac{1}{\sigma^4} \\ 0 & \frac{1}{\sigma^2} & 0 \\ \frac{1}{\sigma^4} & 0 & \frac{2}{\sigma^4} \end{pmatrix}.$$

$$E(U_{1i}V_{2i}^*) = \frac{1}{\sigma^4} \qquad \overline{c} = \frac{1}{2\sigma^4} \qquad E(V_{1i}^2) = \frac{1}{\sigma^2},$$

so

$$Const = -\frac{1}{2}\frac{\frac{1}{\sigma^4}}{\frac{1}{2\sigma^4}\frac{1}{\sigma^2}} = -\sigma^2.$$

Simulations

Second order information correction:

True info

$$
\begin{array}{rcl}
InfoU_2(\hat{\lambda}) & = & InfoU_1 - \left(\boxed{\tfrac{1}{2}\kappa^2} + \boxed{\text{Term 2}} + \boxed{\text{Term 3}} \right). \\
 & & O(n) \quad - \quad O(1)
\end{array}
$$

Oneway ANOVA

Four pivotals:

1.
$$\frac{\sqrt{n\,p}(\hat{\psi}-\psi)}{\sqrt{2}\psi}$$

2.
$$\frac{\sqrt{n\,p}(\hat{\psi}-\psi)}{\sqrt{2}\psi}-\frac{1}{\sqrt{2}}$$

3.
$$\frac{\sqrt{n\,p}(\tilde{\psi}-\psi)}{\sqrt{2}\psi}$$

4.
$$a(\kappa,i)\frac{\sqrt{n\,p}(\tilde{\psi}-\psi)}{\sqrt{2}\psi}$$

$$a(k,i)=1-\frac{\kappa^2}{4\,n\,i}$$

Exact 95% coverage rates

$n = p$	1	2	3	4
2	0.9915	0.9770	0.9080	0.9420
3	0.9866	0.9737	0.9254	0.9511
4	0.9533	0.9730	0.9271	0.9547
5	0.9355	0.9716	0.9298	0.9538
10	0.9089	0.9620	0.9388	0.9510
15	0.9021	0.9580	0.9424	0.9504
20	0.8991	0.9559	0.9443	0.9502

Interpretation

$$\hat{\psi} = \frac{1}{p}\frac{1}{n}\sum(x_{ij} - \overline{x}_i)^2$$

$$\tilde{\psi} = \frac{1}{p}\frac{1}{n-1}\sum(x_{ij} - \overline{x}_i)^2$$

If use information correction then get

$$a(\kappa, i)\sqrt{np}\,(\tilde{\psi} - \psi) = \sqrt{(n-1)p}\,(\tilde{\psi} - \psi).$$

Exponential family, Non-canonical links

- Issue: conditional structure disappears.
- Projected scores can be calculated.
- Must have symbolic math package.

Extension of the binary matched pairs.

X_i and $Y_i, i = 1, \cdots, p$ independent binomial r.v's, based on $n = p$ trials.

Success probabilities:

$$q(0, \lambda_i) \qquad \text{and} \qquad q(\psi, \lambda_i).$$

Take q to be complementary log-log link,

$$q(\psi, \lambda_i) = 1 - e^{-\exp(\psi + \lambda_i)}.$$

1000 replicates at each level of p, $p = 10, \cdots, 20$.

$\lambda_i = -2.0$, $\psi = 2.75$.

Table 2: Number of coverage failures from

1000 replicates of 90% confidence intervals for ψ

Trials, n = p	$\hat{\psi}$ Failures	$\tilde{\psi}$ Failures
10	160	70
11	187	108
12	169	103
13	161	99
14	165	106
15	156	104
16	162	102
17	149	93
18	177	117
19	159	102
20	170	107

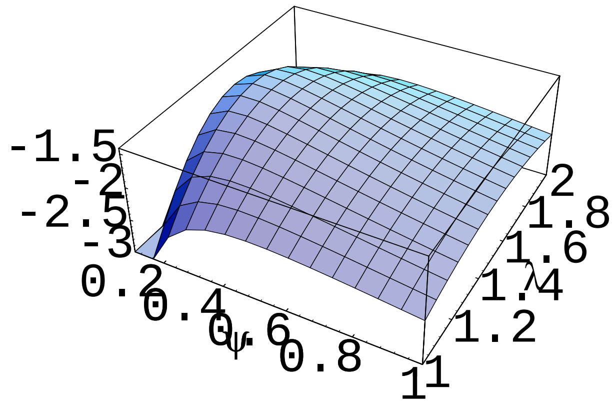
Circular Normal problem

(X,Y) come from a bivariate normal with mean lying on a circle with unknown radius ρ.

Simulated 95% coverage rates

n = p	1	2	3	4
2	95052	95831	93715	94483
3	94693	95420	94287	94801
4	94597	95326	94568	94945
5	94586	95281	94668	94988
10	94348	95119	94834	94986
20	94322	95142	95007	95074

Maximizing the log-likelihood



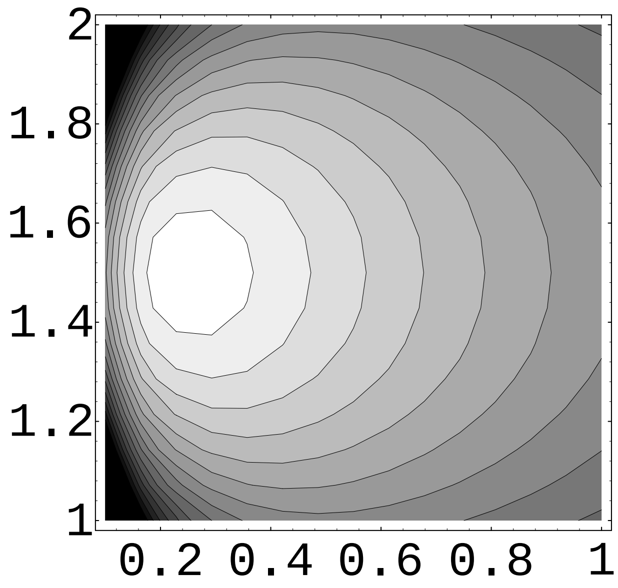


Figure 1: Normal variance and mean