# Statistics 001.

## Final Exam, Fall 1999

This is a closed book examination. You may bring in a formula sheet (1 page). Answer all questions in the blue book. Work all numerical answers out completely.

**Question 1.** An individual is saving for their child's education. They have determined they need to have \$100,000 available 10 years from today.

(a) 10pts. If they were to make a single lump sum investment today, in a fund that guaranteed a nominal yearly return of 7.5% per annum (on a continuously compounded basis) how much would they be required to deposit right now?

$$P_0 e^{r\,t} = 100,000.$$

$$P_0 = 100,000 e^{-r\,t}.$$

$$P_0 = 100,000 e^{-0.075 \times 10}.$$

$$P_0 = 47,237.$$

(b) 5pts. In 5 years, at what rate is their money increasing?

$$\frac{d}{dt} P_0 e^{r\,t} = P_0 \times r \times e^{r\,t} = 47,237 \times 0.075 \times e^{0.05 \times 5} = 5154.7.$$

It is growing at a rate of rougly \$5000 per year.

**Question 2.**

The number of viewers of a television series is approximated by the function

$$N(x) = (60 + 2\,x)^{\frac{2}{3}} \qquad (1 \le x \le 26),$$

where $N(x)$ measured in millions, denotes the number of viewers in the x-th week.

(a) 5pts. How many people watched the series in the 4th week?

$$(60 + 2 \times 4)^{\frac{2}{3}} = 16.66,$$

so 16.66 million watched.

(b) 5pts. At what rate was the number of viewers changing in the 4th week?

$$\frac{d}{dx}(60 + 2\,x)^{\frac{2}{3}} = \frac{2}{3}(60 + 4\,x)^{-\frac{1}{3}} \times 2 = 0.33,$$

so it was growing at a rate of one third of a million viewers per week.

(c) 10pts. Based only on your answers to parts (a) and (b) approximate the number of viewers in the 6th week. Compare this to the number of viewers that the formula predicts in the 6th week.

$$N(6) \sim N(4) + 2 \times \frac{d}{dx} N(x)\Big|_{x=4} = 16.66 + 2 \times 0.33 = 17.33.$$

The actual number was N(6) = 17.31, so it's a pretty good approximation.

**Question 3.** A company has found that their costs of production are determined predominantly by labor hour costs according to the relationship

$$C = 1000 + 15\,H,$$

where $C$ are the costs, and $H$ denotes the number of labor hours purchased.

(a) 2pts. What is the mathematical name given to this type of relationship?

Linear

(b) 3pts. Interpret the components of the cost equation.

The 1000 represents a fixed cost and the 15 indicates that the wage rate is $15/hr.

The number of labor hours is related to the number of units produced, $x$, through the relationship

$$H = 30 + 20\,x + 300\,ln(x + 1).$$

(c) 5pts. Find an expression for the marginal cost of production.

$$\frac{dC}{dx} = \frac{d}{dx}\left(1000 + 15 \times (30 + 20\,x + 300\,ln(x+1))\right) = 300 + 4500\frac{1}{x+1}.$$

(d) 10pts. When 100 units are produced, what is the elasticity of costs with respect to the number of units produced? Interpret the elasticity.

$$\frac{dC}{dx} \times \frac{x}{C} = \left(300 + 4500\,\frac{1}{x+1}\right) \times \frac{x}{(1000 + 15 \times (30 + 20\,x + 300\,ln(x+1)))}.$$

Evaluating this at $x = 100$ gives 0.66, so a 1% change in output gives rise to a 0.66% change in costs.

**Question 4.**

A transit authority is considering whether or not to hike fares in an attempt to increase revenues. Currently 6000 people on average take the train each day and pay \$1.50 per ride.

A study has indicated that for each 25 cent increase in fare, the number of riders will decrease by 1000 per day.

(a) 5pts. Write down an expression for the relationship between the cost of the fare expressed as a function of the number of expected riders; call the fare cost $p$ and the number of riders $x$.

This is a linear relationship, we know that the pair $(p = 1.5, x = 6000)$ is on the line, so we need only one other point to identify the line. If the fare increases by 50 cents, then ridership will decrease to 4000, so the point $(p = 2, x = 4000)$ is also on the line. The line formula is

$$p = c + m\,x,$$

so we need to find the intercept, c, and the slope, m. We know that

$$1.5 = c + m \times 6000,$$

and

$$2.0 = c + m \times 4000.$$

Solving these equations for c and m, indicates that $m = -\frac{1}{4000}$, and $c = 3$, so the equation is

$$p = 3 - \frac{1}{4000}\,x.$$

(b) 5pts. Write down an expression for the revenue (number of riders times fare cost) that would be associated with any particular number of riders.

$$\text{Revenue} = p\,x = \left(3 - \frac{1}{4000}\,x\right)x = 3\,x - \frac{x^2}{4000}.$$

(c) 10pts. What level of ridership maximizes revenue?

$$\frac{dR}{dx} = 3 - \frac{x}{2000}.$$

Setting equal to 0 gives $x = 6000$, and this provides a maximum by noting that the second derivative is negative.

(d) 5pts. Should the transit authority change the cost of a fare?

No, because 6,000 is the current level of ridership.

**Question 5.**

A rumor circulated on campus according to the equation

$$R(t) = \frac{1000}{1 + 199e^{-0.8t}},$$

where $R(t)$ indicates the number of people who had heard the rumor $t$ days after it was first heard.

(a) 5pts. How many people had heard the rumor after the first day?

$$R(1) = \frac{1000}{1 + 199e^{-0.8\times 1}} = 11.$$

(b) 5pts. How many people eventually heard the rumor?

$$\lim_{t\to\infty} R(t) = 1000.$$

(c) 10pts. How long was it until half of those people to hear the rumor had heard it?

$$500 = \frac{1000}{1 + 199e^{-0.8\times t}}.$$

$$2 = 1 + 199e^{-0.8\times t}.$$

$$1/199 = e^{-0.8\times t}.$$

$$-ln(199) = -0.8 \times t.$$

$$-ln(199)/-0.8 = t.$$

$$6.61 = t,$$

so it took about 7 days.