Statistics 002.

Mock Final Exam Solutions, Spring 2000

This is a closed book examination. You may bring in a formula sheet (1 page). Answer all questions in the blue book. Work all numerical answers out completely.

Question 1.

The percentage return on a particular stock has been found to be well approximated by a random variable, X, with the following pdf:

$$f(x) = k(100 - x^2),$$

for x between -10 and +10.

(a) 5pts. What value must k be to make this a valid pdf?

Must check that the pdf is greater than or equal to 0, and the area under the pdf is one. The area is

$$\int_{-10}^{+10} k(100 - x^2)dx,$$

which equals $4000\frac{k}{3}$, so that k must be $\frac{3}{4000}$.

(b) 5pts. What is the probability that X is less than 5%?

$$\int_{-10}^{-5} \frac{3}{4000}(100 - x^2)dx = 0.15625.$$

(c) 5pts. Find the mean of X.

Note that this is a symmetric distribution about 0, so the mean must be zero to.

(d) 5pts. Find the variance and standard deviation of X.

$$Var(X) = \int_{-10}^{+10} \frac{3}{4000}(x - \mu)^2(100 - x^2)dx,$$

where μ is the mean and is in fact 0 (using part c). Var(X) = 20, and the standard deviation is the square root of the variance, that is $\sqrt{20} = 4.472$.

Question 2.

The weekly revenues, X, of a supermarket have been found to follow a normal distribution with the following pdf

$$f(x) = \frac{1}{1000\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-10000}{1000}\right)^2},$$

(a) 5pts. Write down the mean and standard deviation of X.
Mean = 10000, standard deviation = 1000.
(b) 5pts. Find the probability that revenues exceed \$12,500.
$P(X > 12500) = P(Z > 2.5) = 0.006.$
(c) 5pts. Find the probability that revenues are between \$8,000 and \$9,500.
$P(8000 < X < 9500) = P(-2 < Z < -0.5) = 0.286.$
(d) 5pts. Find the critical revenue value, V, such that there is only a 5% chance of getting a weekly revenue less than V.

This is one of those questions in which you have to use the tables "backwards", that is you are given the probability, from which you have to figure out the critical value on the Z-scale, and then the critical value on the original scale.
We have

$$P(Z < Z_{crit}) = 0.05,$$

so that $Z_{crit} = -1.645$.
This means that $\frac{V-\mu}{\sigma} = -1.645$, so that $V = \mu - 1.645\sigma$; plugging in μ and σ gives $V = 10000 - 1.645 \times 1000 = 8355$.

Question 3.
The average daily high temperature in Philadelphia is approximated by the function

$$f(t) = 63 + 23\, sin(2\pi(t - 100)/365),$$

where t indicates the number of days into the year. (t = 1, means Jan 1, t = 34 means Feb 3 etc.)
(a) 5pts. Sketch the temperature curve for a year and find the expected high temperature on Jan 1.
Plug in $t = 1$ to the equation to get 40.2
(b) 5pts. Find the value of t which gives the hottest day.
This ask you to find the max of the function. Differentiating gives,

$$f'(t) = 23 \times (2\pi/365) cos(2\pi(t - 100)/365),$$

this equals 0 when

$$cos(2\pi(t - 100)/365) = 0,$$

that is when the expression inside the bracket equals either $\frac{\pi}{2}$ or $\frac{3\pi}{2}$. So we have

$$2\pi(t - 100)/365) = \frac{\pi}{2},$$

so

$$2(t - 100)/365) = \frac{1}{2},$$

$$(t-100) = \frac{365}{2} \times \frac{1}{2},$$
$$t = 100 + \frac{365}{2} \times \frac{1}{2},$$

that is $t = 191.25$.
Note that the other equation would give you the coldest day of the year!

(c) 5pts. At what rate is the temperature increasing on t = 60? This says find the derivative at t = 60. We know from part b that

$$f'(t) = 23 \times (2\pi/365)cos(2\pi(t-100)/365),$$

plugging in $t = 60$ gives $f'(60) = 0.306$, or 0.3 degrees/day.

(d) 5pts. What is the value of t such that it is getting colder, quickest?

In English, this question is asking when is the derivative function (getting colder), itself minimized.

$$f''(t) = -23 \times (2\pi/365)^2 sin(2\pi(t-100)/365).$$

This equals 0 when

$$sin(2\pi(t-100)/365) = 0$$

This happens when the expression in the parenthesis equals either 0 or π. That is either at $t = 100$ or $(t-100)/365 = \frac{1}{2}$. So that $t = 100$ or $t = 282.5$. Clearly the second of these is the desired solution, as it is in the fall whereas $t = 100$ is in the spring.

Question 4.

A company's profit is given as a function of their output x by

$$p = -500 + 75\, ln(x),$$

for x between 200 and 1000.

(a) 10pts. The profit at $x = 900$ is 10.18. Use a two term Taylor series about the point $x = 900$ to approximate the profit at $x = 910$. Compare the Taylor series approximation to the true value and comment on the accuracy of the approximation.

$$p_2(x) = p(900) + f'(900)(x-900) + f''(900)\frac{1}{2!}(x-900)^2.$$
$$f'(x) = \frac{75}{x} \qquad f''(x) = -\frac{75}{x^2},$$

so

$$p_2(x) = p(900) + \frac{75}{900}(x-900) - \frac{75}{900^2}\frac{1}{2!}(x-900)^2.$$

Taylor series approximation for $x = 910$ is

$$p_2(910) = p(900) + \frac{75}{900}(910 - 900) - \frac{75}{900^2}\frac{1}{2!}(910 - 900)^2 = 11.0831.$$

The actual value is $p(910) = 11.0834$, in other words it's an excellent approximation.

(b) 10pts. At what value of x does the company break even? Use the Newton-Raphson algorithm with a starting value of $x = 900$, and take 3 steps.

Note that the question is asking you at which value of x Profit is zero, that is finding the root of the profit function.
First calculate the Newton-Raphson update rule:

$$x_{new} = x_{old} - \frac{f(x_{old})}{f'(x_{old})}.$$

Here, $f'(x) = \frac{75}{x}$, so that

$$x_{new} = x_{old} - \frac{-500 + 75ln(x_{old})}{\frac{75}{x_{old}}} = x_{old} - \frac{x_{old}}{75} \times (-500 + 75ln(x_{old})).$$

Now plug in $x_0 = 900$, to get $x_1 = 777.8447$.
Similarly $x_2 = 785.7319$, and $x_3 = 785.7720$.

Question 5.

The number of car accidents that an individual has in a year has been determined to follow a Poisson distribution with mean equal to 0.25.

(a) 5pts. What is the probability that an individual has no accidents in a year?
$P(X = 0) = \frac{e^{-0.25}0.25^0}{0!} = 0.7788.$

(b) 5pts. What is the probability that an individual has three or more accidents in a year?
$P(X \geq 3) = 1 - P(X \leq 2) = 1 - (\frac{e^{-0.25}0.25^0}{0!} + \frac{e^{-0.25}0.25^1}{1!} + \frac{e^{-0.25}0.25^2}{2!})$. On calculation this gives $1 - (.7788 + .1947 + .0243) = 0.0022$.

(c) 10pts. If I define an individual as a "bad risk" if they are in the worst 1% of drivers in terms of the number of accidents they have had in a year, then how many accidents in a year defines a "bad risk"?

This is one of those questions where you have been given the probability (0.01) and you have to find the X. It ask to find the smallest x_{crit} such that $P(X \geq x_{crit} < 0.01)$. By looking at the answer to part (b), we see that an individual with three or more accidents is certainly in the worst 1% of drivers (because there is only a 0.0022 probability of being this bad). The chances of 2 or more accidents is 0.0265, so that the "2 or more" gives more than a 1% chance of being defined as a bad driver, so we have to take the $x_{crit} = 3$ to get a 1% or less chance.