1. Imagine an experiment where you flip a biased coin where the probability of heads is 1/3 until you observe heads for the first time.

(a) What is the sample space of this experiment? How is it different than the sample space where we stop after 8 flips, regardless of whether or not we’ve seen a head?

(b) What is the probability that you observe the first head by the 4th (≤ 4) toss?

(c) What is the probability that it takes more than 2 tosses before you observe the first head?

(d) What is the probability that you see the first head on an even numbered toss?

Solution:

(a) \{H, TH, TTH, TTTH, \ldots\}. If we stop, the space is finite and includes the first elements of the original sample space in addition to TTTTTTT.

(b) 
\[
P(H) + P(TH) + P(TTH) + P(TTTH) = \frac{1}{3} + \frac{2}{3} \times \frac{2}{3} + \left( \frac{2}{3} \right)^2 \times \frac{1}{3} + \left( \frac{2}{3} \right)^3 \times \frac{1}{3}
\]

(c) 
\[
1 - P(H) - P(HT) = 1 - \frac{1}{3} - \frac{2}{3} \times \frac{2}{3}
\]

(d) 
\[
\frac{1}{3} \sum_{i=0}^{\infty} \left( \frac{2}{3} \right)^{2i+1} = \frac{1}{3} \times \frac{2}{3} \sum_{i=0}^{\infty} \left( \frac{4}{9} \right)^i = \frac{2}{9} \times \frac{1}{5/9} = \frac{2}{5}
\]

2. Explain what is wrong with the following reasoning:

I have an experiment where I flip a coin and roll a six-sided die. Since the coin must come up heads or tails, \(P\{H, T\} = 1\) and since the die must show a number from 1 to 6, I must have \(P\{1, 2, 3, 4, 5, 6\} = 1\) as well. Since these events are disjoint, I have 
\[
P\{\{1, 2, 3, 4, 5, 6\} \cup \{H, T\}\} = P\{\{H, T\}\} + P\{\{1, 2, 3, 4, 5, 6\}\} = 2
\]

Solution: We didn’t correctly define our sample space. Since we are doing two things, the sample space needs to describe both. It should be \{(1, H), \ldots (6, H), (1, T), \ldots, (6, T)\}

3. Let \(A\) and \(B\) be events such that \(P(A) = 3/4\) and \(P(B) = 1/3\).
(a) Show that \( \frac{1}{12} \leq P(A \cap B) \leq \frac{1}{3} \).

(b) Find similar bounds for \( P(A \cup B) \).

Solution:
(a) The largest the intersection can be is \( \Omega \), so we would have
\[
P(A \cup B) = P(\Omega) = 1 = P(A) + P(B) - P(A \cap B) = \frac{13}{12} - P(A \cap B)
\]
Then we just solve for \( P(A \cap B) \). For the lower bound, note that we achieve it when \( B \subseteq A \)
(b) \( 3/4 \) and \( 1 \).

4. (a) Let \( P(A) = 0.55 \), \( P(B^C) = 0.35 \) and \( P(A \cup B) = 0.75 \). Find \( P(B) \) and \( P(A \cap B) \)
(b) Let \( P(A^C) = 0.6 \), \( P(B) = 0.3 \) and \( P(A \cap B) = 0.2 \). Find \( P(A \cup B) \)
(c) Let \( P(A) = 0.5 \), \( P(B) = 0.2 \), \( P((A \cup B) \cap C^C) = 0.6 \) and \( P(B \cup C) = 0.3 \). If \( A \) and \( B \) are disjoint, and \( A \) and \( C \) are disjoint, what is \( P(C) \)?

Solution
(a) \( P(B) = 0.65 \). \( P(A \cap B) = -P(A \cup B) + P(A) + P(B) = 0.45 \)
(b) \( P(A) = 0.4 \). \( P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.5 \)
(c) \( P((A^C \cap B^C) \cup C) = 1 - P((A \cup B) \cap C^C) = 0.4 \)

5. You are competing in a tournament in which you play three games and win if you win two in a row. There are two opponents, one good and one bad, call them \( B \) and \( G \). You must play each one at least once, you can pick the order, but you cannot play two in a row against the same person. So you can play either \( BGB \) or \( GBG \). If the probability of winning against \( B \) is \( p \) and the probability of winning against \( G \) is \( q \), where \( p < q \), which choice is better? Why?

Solution: The question was supposed to have \( q < p \). But either way you get the same idea. There are two ways to win: either win the first two matches or lose the first and win the last two. So, if you play the good opponent in the middle and the bad one twice on the ends, the probability of winning is
\[
kp + (1 - p)qp
\]
If you play the good one twice on the ends, the probability of winning is
\[
qp + (1 - q)pq
\]
Since \( 1 - q > 1 - p \) we want the play the good opponent twice. Intuitively, the middle game is the most important, and we must win that one to win the match. So, we want the highest change to win there, otherwise we don’t get a second chance in the third game no matter what.

6. I’m trying to send you a message in binary. But the channel I’m sending it through sometimes corrupts the bits. Let \( S_0, S_1 \) be the events that I send a 0 or 1 respectively, and \( R_0, R_1 \) be the events that you receive a 0 or 1 respectively. Assume you know that \( P(R_1 | S_1) = 0.95 \), and \( P(R_0 | S_0) = 0.98 \). Let \( p = P(S_0) \) and \( 1 - p = P(S_1) \).
(a) What is the probability of receiving a 1?
(b) Given that you read a 1, what is the probability I sent a 1?

(c) What is the overall probability of receiving the correct bit?

(d) (Hard) Suppose now that I’m sending you two bit messages and that you received 11. If you know that the probability of my sending a 1 following another 1 is 0.01, what is the probability I actually sent 01. Assume that, in the absence of any additional information, the probability of a 1 as the first or second bit in my message is the same

Solution:

(a) \( P(R_1) = P(R_1 | S_1) P(S_1) + P(R_1 | S_0) P(S_0) = \frac{1}{2}(0.95 + 0.02) \)

(b) \( P(S_1 | R_1) = \frac{P(R_1 | S_1)P(S_1)}{P(R_1)} \)

(c) \( P((R_1 \cap S_1) \cup (R_0 \cap S_0)) P(R_1 | S_1) P(S_1) + P(R_0 | S_0) P(S_0) = 0.965 \)

(d) \( P(S_{01} | R_{11}) = \frac{P(R_{11} | S_{01}) P(S_{01})}{P(R_{11})} \)

\( P(R_{11} | S_{01}) = 0.02 \times 0.95 \) since corruption of bits is independent. \( P(S_{01}) = \frac{11}{22} \) since seeing a 0 tells us nothing about the next bit.

\( P(R_{11}) = P(R_{11} | S_{11}) P(S_{11}) + P(R_{11} | S_{01}) P(S_{01}) + P(R_{11} | S_{10}) P(S_{10}) + P(R_{11} | S_{00}) P(S_{00}) \)

Then, \( P(S_{11}) = \frac{1}{2} \times 0.01 \), \( P(S_{10}) = \frac{1}{2} \times 0.99 \). The other two are just 1/4

7. (a) Suppose that \( A, B \) are events and \( A \subseteq B \). Can \( A \) and \( B \) be independent?

(b) Let \( A, B, \) and \( C \) be events that are all independent. Show that \( A \) and \( B \cup C \) are independent.

(c) If \( A \) and \( B \) are independent, show that \( A \) and \( B^C \) are independent. (Hint: Use the facts that \( B \) and \( B^C \) are disjoint and that \( P(B) = 1 - P(B^C) \))

(d) If \( A \) and \( B \) are independent, and \( B \) and \( C \) are independent, are \( A \) and \( B \cup C \) independent?

Provide a proof or counterexample.

Solution:

(a) \( P(A \cap B) = P(A) \). They would only be independent if \( P(A) = 0 \). Simply saying “no” is OK here if you assume \( P(A) > 0 \)

(b) \( P(A \cap (B \cup C)) = P((A \cap B) \cup (A \cap C)) = P(A \cap B) + P(A \cap C) - P(A \cap B \cap C) \)

Then use independence and the above simplifies to

\( P(A)(P(B) + P(C) - P(B \cap C)) \)

(c) \( P(A) = P(A \cap B) + P(A \cap B^C) \) since \( B \) and \( B^C \) form a partition. By independence \( P(A \cap B) = P(A)P(B) \) so \( P(A) - P(A)P(B) = P(A \cap B^C) \). Factoring the left side gives \( P(A)(1 - P(B)) = P(A)P(B^C) / \)

(d) Let \( A = C \) or most other examples where \( A \) and \( C \) are dependent.
8. (a) Let $A$ be the event corresponding to “A couple has three children and the oldest is a boy.” and $B$ be the event described by “A couple has three children and at least one is a boy.” Write out $A$ and $B$.

(b) Let $G$ be the event “Their youngest child is a girl.” Compute $P(G \mid A)$ and $P(G \mid B)$. Is this intuitive?

Solution:
(a) $A = \{BBB, BGB, GBB, GGB\}, B = \{BGG, GBG, GGB, GBB, BGB, BBG, BBB\}$

(b) $P(G \mid A) = \frac{1}{2}, P(G \mid B) = \frac{3}{7}$

In casual speech, the two events sound the same, but I’m actually giving you different information. In event $A$ I completely specify one of the children’s sexes. But in event $B$ I only eliminate one possibility of the 8 total.

9. Let $Q(A) = P(A \mid B)$ for $A, B \in \Omega$ where $\Omega$ is some sample space. Show, using the definition of conditional probability, that $Q$ satisfies the axioms of probability.

Solution: Assume $P(B) > 0$.

(a) $P(A \mid B) = \frac{P(A \cap B)}{P(B)} \geq 0$ since it is a ratio of two nonnegative quantities

(b) $P(\Omega \mid B) = \frac{P(\Omega \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$

(c) If $A \cap C \emptyset$, then so are $(A \cap B) \cap (C \cap B) = \emptyset$ as well. This implies $P(A \cup C \mid B) = P(A \mid B) + P(C \mid B)$

Extra Credit: Suppose you have events $A_i$ where $i \in \{1, \ldots, n\}$. You are told that at least one event must occur, so $P(\cup_{i=1}^{n} A_i) = 1$, and that no more than two of them may occur together, so $P(A_i \cap A_j \cap A_k) = 0$ when $i \neq j \neq k$. Suppose further, that $P(A_i) = p$ for all $i$ and $P(A_i \cap A_j) = q$ for all $i \neq j$. Show that $p \geq 1/n$ and $q \leq 2/n$. To make this a bit easier, start with $n = 2$, then $n = 3$ and so on. Then see how it all generalizes.