YOUR NAME

INSTRUCTIONS: No notes, no calculators, and no communications devices are permitted. Please keep all materials away from your desk. You if you need scratch paper use the back of the pages on the test. You should not have any blue books or stray papers near your desk.

You should be attentive to clear presentation of your answers. You are encouraged to write as neatly as you can. You are also encouraged to use any extra time you have to check your work for accidental mistakes.

The problems have equal weight, but, within a given problem, the parts typically have unequal weights. There may be some simple sort answer parts, and another parts may require a more substantial calculation. The parts receive weight based on the relative effort required for the part.

You are expected to adhere strictly to the Academic Code of Ethics of the University of Pennsylvania.

Leave the table below BLANK. Be sure to Print and Sign your name above.

<table>
<thead>
<tr>
<th>Problem Number</th>
<th>Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td></td>
</tr>
<tr>
<td>2.</td>
<td></td>
</tr>
<tr>
<td>3.</td>
<td></td>
</tr>
<tr>
<td>4.</td>
<td></td>
</tr>
<tr>
<td>5.</td>
<td></td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td></td>
</tr>
</tbody>
</table>
Problem 1. Consider a Markov chain \( \{X_n : n = 0, 1, \ldots \} \) with the state space \( S = \{1, 2\} \) and the transition matrix
\[
p = \begin{pmatrix}
1/3 & 2/3 \\
1/2 & 1/2
\end{pmatrix}.
\]
In the space below, answer the following questions:

1. Find a stationary distribution for this chain.
2. Find the value of the probability below. Leave your answer in factored form; i.e. do not “simplify” your answer.
   \[
P(X_1 = 1, X_2 = 2, X_3 = 1, X_4 = 1 | X_0 = 1).
\]
3. If \( n \) is an integer, what is the probabilistic interpretation of \( p^n \), the \( n \)'th power of the transition matrix? Specifically, what is the meaning of the individual components of this matrix?
4. For \( x \in S \), we let \( T_x = \min \{ n \geq 1 : X_n = x \} \) as usual. For the Markov chain in this problem, what is the numerical values of \( E_1[T_1] \) and \( E_2[T_2] \)? Here, as always, \( E_x[T_y] \) denotes the expected value of \( T_y \) given that \( X_0 = x \).

Note: This requires virtually no calculation beyond what you have already done.
Problem 2. Consider random walk on $S = \{0, 1, 2, 3, 4, 5\}$ with “hard” reflection. Specifically, we have $1/2 = p(x, x + 1) = p(x, x - 1)$ for $x \in \{1, 2, 3, 4\}$ and for the end points we have $p(0, 1) = p(5, 4) = 1$.

(1) Find the stationary distribution for this Markov chain.
(2) Suppose you start in state 4 and you move according to the transition probabilities until you either hit state 0 or state 5 at which time you stop and receive an award. If you stop in state 0 your reward is 5 dollars and if you stop in state 0 your reward is 20 dollars. What is the expected value of your award.
Problem 3. The classical enlisted ranks in the U.S. Army are private, corporal, and sergeant — though there is a more sophisticated system of ranks in the modern army. Identify these ranks with the states \{1, 2, 3\} of a Markov chain and assume that the chain has the month-to-month transition matrix

\[
p = \begin{pmatrix}
0.8 & 0.2 & 0 \\
0.2 & 0.6 & 0.2 \\
0 & 0 & 1
\end{pmatrix}
\]

Thus, for example, the probability that a corporal is “busted” back to private is 0.2 and the probability of being promoted to sergeant is also 0.2, but once a “Sarge” always a Sarge.

(1) Guess the stationary distribution for this Markov chain and do the calculation needed to confirm your guess.

(2) Suppose \(X_0 = 1\), so the initial rank is private. Let \(W\) be the first time when the chain arrives at state 3, i.e. the private eventually makes it to sergeant. Calculate the expected value of \(W\). That is, calculate the expected number of months until a private becomes a sergeant, under this model.
Problem 4. Consider the set \( V = \{1, 2, 3, 4\} \) and view these states as the corners of a square. Now define a graph by taking the edges of the square together with the diagonals. Formally, we have an edge between \( x \) and \( y \) for all of the pairs

\[(1, 2), (2, 3), (3, 4), (4, 1), (1, 3), \text{and} (2, 4).\]

This is called regular graph since each vertex has the same degree; in this case each vertex has degree 3. Let \( \{X_n : n = 0, 1, \ldots\} \) be the random walk on this graph. If you are at vertex \( x \) then on the next transition, if \( y \) is one of the three neighbors of \( x \), then you go to \( y \) with probability \( \frac{1}{3} \).

1. Guess the stationary distribution for this Markov chain and confirm your guess with a calculation.
2. Let \( V_x = \min\{n \geq 0 : X_n = x\} \). Calculate \( E_1[T_3] \), that is, calculate the expected number of steps to get from state 1 to state 3. You can simplify your work by being attentive to any symmetries that exist in the problem.
3. Now modify the chain by adding a loop with probability \( \frac{1}{2} \) at each vertex, so if you are at \( x \) you move to \( x \) with probability \( \frac{1}{2} \) and you move to any one of the three old neighbors with probability \( \frac{1}{6} \). Now, use your gambler’s good sense to guess the expected time that it takes to get from vertex 1 to vertex 3. Give your intuitive reasons for your guess. You don’t have to give a formal confirmation.
Problem 5. Consider the simple random walk \( \{X_n : n = 0, 1, \ldots \} \) on the set of all positive and negative integers (including 0). That is one has
\[
p(x, x + 1) = p(x, x - 1) = 1/2
\]
for all \( x \in \{ \ldots, -2, -1, 0, 1, 2, \ldots \} \).

(1) Take \( N \) to be a fixed non-zero integer and let
\[
W = \min \{ n \geq 0 : X_n = N, \text{ or } X_n = -N \}.
\]
What is the value of \( E_0[W] \)? You can use any facts you know to do this calculation; you do not have to rederive any formulas — just use what you know.

(2) Now consider the set of times \( \{ 1 \leq n \leq W : X_n = 0 \} \). Let \( Z \) be the cardinality of this set; that is, \( Z \) is the number of times \( n \geq 1 \) at which the walk visits state 0 before hitting either state \( N \) or state \( -N \). Calculate the expected value of \( Z \).

Note: We have not solve a problem before that has precisely this form, but you do have the tools to solve this problem. You just need to think clearly. You should think about first step analysis which we have used many times. You should think about any facts that you know that can help you to simplify the problem. Before you get lost in a calculation, you should think about a good approach. Done well, the calculation is rather brief. Whenever you think you have an answer, think of some “stress tests” to see if your answer makes sense.