last problem outlines how the Black–Scholes PDE (and its derivation) may be modified to account for the payment of stock dividends.

10.1 (Calculation of Delta — First and Finest of the Greeks).

The Black–Scholes formula for a European call option is often best written as a three part formula:

\[
f(t, S) = S\Phi(D_+) - e^{-\tau r}K\Phi(D_-),
\]

where the arguments of the normal distribution \( \Phi \) are given by

\[
D_+ = \frac{\log(S/K) + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \quad \text{and} \quad D_- = \frac{\log(S/K) + (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}},
\]

As usual, \( S \) is the current stock price, \( K \) is the strike price, \( \sigma \) is the volatility parameter of the stock model (10.3), and \( \tau = T - t \) is the time until expiration of the option. The various derivatives of \( f \) with respect to these parameters now have standard names that are given by Greek letters. In aggregate, these derivatives are known simply as “the Greeks.”

The most important of the Greeks is \( \Delta \), the derivative of \( f \) with respect to stock price. Naturally, \( \Delta \) tells us about the sensitivity of an option price to the changes in the stock price, but its greatest value comes through the stock weight formula (10.9) which tells us that \( \Delta \) is size of the stock position in the replicating portfolio.

Prove that \( \Delta(S, K, r, \sigma, \tau) \) has the surprisingly simple formula:

\[
\Delta(S, K, r, \sigma, \tau) \equiv \frac{\partial f}{\partial S} = \Phi(D_+).
\]  

(10.20)

One might mistakenly “prove” this formula by ignoring the fact that \( D_+ \) and \( D_- \) depend on \( S \). Since they do depend on \( S \), an honest proof of the identity (10.20) calls for the chain rule plus a little work. This work can be made more pleasant by taking advantage of the factorization of \( D_+^2 - D_-^2 \).

Solution for Problem 10.1. Honest calculus tells us that

\[
\frac{\partial f}{\partial S} = \Phi(D_+) + S\phi(D_+\frac{1}{S\sigma\sqrt{\tau}} - e^{-\tau r}K\phi(D_-\frac{1}{S\sigma\sqrt{\tau}}),
\]

so we have to show the last two terms cancel. That is, we need to show

\[
S\phi(D_+) = e^{-\tau r}K\phi(D_-),
\]

where \( \phi \) is the normal density. Now, when we take logs and collect terms, we see this boils down to showing

\[
\log(S/K) + \tau r = (D_+^2 - D_-^2)/2.
\]  

(10.21)

We have the factorization \( D_+^2 - D_-^2 = (D_+ + D_-)(D_+ - D_-) \) and substitution from the definition of \( D_+ \) and \( D_- \) then gives us
\[
(D_+ + D_-)(D_+ - D_-) = \frac{1}{\tau \sigma^2} 2(\log(S/K) + \tau r)(\tau \sigma^2),
\]
so we do indeed have the equality (10.21). □

10.2 (Gamma: Measuring How Delta Changes).

The Gamma of an option is the second derivative of the option price with respect to the price of the underlying security. Equivalently, Gamma is the derivative of Delta with respect to stock price. The importance of Gamma is that in a world where Delta may be used to design a hedge, Gamma tells you how quickly that hedge may drift out of whack.

(a) Find a formula for \( \Gamma \equiv \partial^2 f / \partial^2 S \equiv \partial \Delta / \partial S \) for a European call option under the Black-Scholes model.

(b) Consider two options on the same stock, both “at the money” (i.e. \( K = S \)), one with a year to expiry, one with two years to expiry. Determine which option has the smaller \( \Gamma \) (and consequentially which is easier to hedge).

Solution for Problem 10.2. Since \( \partial D_+ / \partial S = 1 / (S \sigma \sqrt{\tau}) \), the chain rule and our formula (10.20) for \( \Delta \) give us

\[
\Gamma = \phi(D_+)/(S \sigma \sqrt{\tau}),
\]

which is a useful answer for Part a. For an at the money option, we have \( D_+ = (r + \sigma^2/2) \sqrt{\tau}/\sigma > 0 \), increases with \( \tau \). Thus, for such options, both factors in our formula (10.22) decrease as \( \tau \) increases. Hence, the two year option has the smaller \( \Gamma \) and is easier to hedge.

10.3 (Equal Gammas for Puts and Calls). Consider a European put and a European call on the same underlying stock and with the same strike price. Explain why — even without assuming the Black-Scholes model — one has the identity:

\[
\Gamma_{\text{call}} = \Gamma_{\text{put}}.
\]

Solution for Problem 10.3. The formula (10.2) for put-call parity, we have \( C - P = S - e^{-r\tau} K \). Taking one \( S \)-derivative of this formula gives us a striking relationship between the Deltas,

\[
\Delta_{\text{call}} - \Delta_{\text{put}} = 1.
\]

Taking a second derivative shows the equality of the Gammas. Very simple, but very nice.

10.4 (The Rest of the Greeks).

The Greek chorus is filled out by taking the derivatives with respect to interest rate, time, and volatility. These are called respectively Rho, Theta, and Vega; the later not being a Greek letter at all but rather a star in the constellation Lyra (the Harp).
Vega is the most interesting of the lesser Greeks. It provides us with a way to reason about “volatility risk” by telling us how much an option goes up if $\sigma$ goes up, or how much it goes down if $\sigma$ goes down. Ironically, $\sigma$ can’t go anywhere in a pure Black-Scholes world. Of course, in the real world $\sigma$ is just as random as anything else.

(a) Find the simplest formulas you can for Vega for European call options and put options under the Black-Scholes pricing model.

(b) How is the Vega for a put related to the Vega for the corresponding call? Is there a way to see this at a glance?

(c) Theta tells us how much an option will go down just because of the erosion of the time remaining on the contract. Find formulas for Theta for puts and calls in the Black-Scholes world, but before you roll out the calculus, see if you can guess the signs of these derivatives.

(d) Find the simplest formulas you can for Theta for puts and calls in the Black-Scholes world. Is Rho more important for long-term options or for short-term options.

In all of these cases give honest derivations of the formulas you find and be sure to identify any tricky parts of your derivation. For example, in the case of Delta we saw that there is quick but bogus way to get Delta.

This problem may be a little dull since it just involves differentiation and algebra, but it also affords the opportunity to dig into the conceptual meaning of all the features of the Black-Scholes formula. The process of “working out the Greeks” is now a standard (and valuable) part of the development of any option pricing theory.

Solution for Problem 10.4.

Let $C$ and $P$ denote the call price and the put price respectively:

$$\frac{\partial C}{\partial \sigma} = S\phi(D_+)\sqrt{\tau} = \frac{\partial P}{\partial \sigma}$$  \hspace{1cm} (10.23)

The equality of the Vegas for puts and calls is immediate from the put-call parity relation $C - P = S - e^{rt}K$, since $\sigma$ does not appear on the right-hand side. For Thetas we are less lucky,

$$\frac{\partial C}{\partial t} = \frac{-S\phi(D_+)\sigma}{2\sqrt{\tau}} - rKe^{-rt}\Phi(D_+) \quad \text{and} \quad \frac{\partial P}{\partial t} = \frac{-S\phi(D_+)\sigma}{2\sqrt{\tau}} - rKe^{-rt}\Phi(D_+).$$

Here we could have guess that both the signs were negative because even in-the-money options lose money if nothing changes except the flow of time. Finally, for Rho we have

$$\frac{\partial C}{\partial r} = K\tau e^{-rt}\Phi(D_-) \quad \text{and} \quad \frac{\partial P}{\partial r} = -K\tau e^{-rt}\Phi(-D_-).$$

Here could one have guessed the signs?
10.5 (A More General Black–Scholes PDE). Consider a model where the stock and bond prices evolve according to

\[
dS_t = \mu(t, S_t)S_t \, dt + \sigma(t, S_t)S_t \, dB_t \quad \text{and} \quad d\beta_t = r(t, S_t)\beta_t \, dt,
\]

where all of the model coefficients \( \mu(t, S_t) \), \( \sigma(t, S_t) \), and \( r(t, S_t) \) are given by bounded functions of the current time and current stock price. One needs an odd economy for the bond rate to depend on a stock’s price, so permitting \( r \) to depend on \( S_t \) is mainly a mathematical amusement. On the other hand, permitting \( \mu(t, S_t) \) and \( \sigma(t, S_t) \) to depend on \( S_t \) and \( t \) has much practical merit.

Consider the coefficient matching method of Section 10.2 and show that the arbitrage price at time \( t \) of a European option with terminal time \( T \) and payout \( h(S_T) \) is given by \( f(T, S_T) \) where \( f(t, x) \) is the solution of the terminal-value problem

\[
\begin{align*}
&f_t(t, x) = -\frac{1}{2} \sigma^2(t, x)xxf_{xx}(t, x) - r(t, x)xf_x(t, x) + r(t, x)f(t, x), \\
&f(T, x) = h(x).
\end{align*}
\]

Finally, show that \( f(t, x) \) and its derivatives may be used to provide explicit formulas for the portfolio weights \( a_t \) and \( b_t \) for the self-financing portfolio \( a_tS_t + b_t\beta_t \) that replicates \( h(S_T) \).

**Solution for Problem 10.5.** The solution of this problem requires no calculation, only a mild epiphany. Examination of the argument Section 10.2 reveals that it may be repeated line by line; one never used the constancy of the model coefficients. Thus, the Back-Scholes PDE holds for the more general model and the analogs of formulas (10.10) and (10.9) for the bond and stock positions remain in force.

10.6 (Black–Scholes PDE and Put–Call Parity).

(a) Consider the contingent claim that corresponds to being long one European call and short one European put on the same stock. Assume also that each option has strike price \( K \) and expiration time \( T \). What is the terminal pay-out function \( h(\cdot) \) that corresponds to this collective claim?

(b) Show by direct substitution into the Black-Scholes PDE (10.11) that \( f^*(t, x) = x - e^{-r(T-t)}K \) is a solution. Use this observation and your answer to part (a) to give an alternative proof of the put–call parity formula (10.2). Is this PDE derivation more general, or less general, than the derivation given at the beginning of the chapter?

**Solution for Problem 10.6.** (a) Here \( h(\cdot) \) would be the line through \((K, 0)\) with slope one, i.e. \( h(x) = x - K \). For part (b) we first evaluate both sides of the Black-Scholes PDE (10.11). We have \( f^*_t(t, x) = -re^{-r(T-t)}K \) and
so the Black-Scholes PDE (10.11) is indeed satisfied by \( f^* \). At the terminal time we also have \( f^*(T, x) = h(x) = x - K \), so, assuming that the solution to the Black-Scholes terminal value problem is unique (as we have been doing so throughout the chapter), we see that \( C - P \), the value of the portfolio of one long call and one short put is equal to \( f_t^*(t, S) = S - re^{-r(T-t)}K \). In other words, we have
\[
C - P = S - re^{-r(T-t)}K.
\]
which is the desired put-call parity formula (10.2).

Naturally, this derivation is much less general than our first one (page 209). The only assumption in that derivation was the existence of a two-way interest rate; in particular, no stochastic model was required for the underlying asset.

\( \square \)

10.7 (Dealing with Dividends). Suppose that we have a stock that pays a dividend at a constant rate that is proportional to the stock price, so that if you hold \( a_t \) units of the stock at time \( t \) then during the time period \([\alpha, \beta]\) your total dividend receipts will be
\[
r_D \int_{\alpha}^{\beta} a_t S_t dt,
\]
where \( r_D \) is a constant which we will call the dividend yield. Suppose we start with a stock that pays no dividend that satisfies the SDE
\[
dS_t = \mu S_t dt + \sigma S_t dB_t
\]
(10.25)
If the dividend policy of the stock is then changed so that the stock pays a continuous dividend (with constant dividend yield \( r_D \)), then funds which would have increased the value of the firm have been distributed to the shareholders. Under the new policy the SDE for \( S_t \) should be changed to
\[
dS_t = (\mu - r_D)S_t dt + \sigma S_t dB_t.
\]
(10.26)
If we assume that we have the usual bond model \( d\beta_t = r\beta_t dt \), then, by analogy with other cases we have studied, one might guess that the change in dividend policy would not lead to a change in the arbitrage pricing PDE. Surprisingly enough, the change in dividend policy does call for a change in the PDE, and it turns out that the appropriate terminal value problem for a European option with time \( T \) payout \( h(S_T) \) is then given by the Merton-Black-Scholes equation:
\[
f_t(t, x) = -\frac{1}{2}\sigma^2 x^2 f_{xx}(t, x) - (r - r_D)x f_x(t, x) + rf(t, x).
\]
Use the coefficient matching method of Section 10.2 to show that this assertion is correct. To solve this problem you will need to make an appropriate modification of the self-financing condition before you start matching coefficients.

**Solution for Problem 10.7.** Since our portfolio also earns cash at rate $r_D$ to be checked and completed from a stock investment with value $a_t S_t$, the self-financing condition is now given by

$$dV_t = a_t dS_t + r_D a_t S_t \, dt + b_t \, d\beta_t.$$  

From the new stock model (10.26) we then have

$$dV_t = \{a_t(\mu - r_d)S_t + b_t r \beta_t + r_D a_t S_t\} \, dt + a_t \sigma S_t dB_t. \quad (10.27)$$

On the other hand, we also have $V_t = f(t, S_t)$ so Itô’s formula gives

$$dV_t = f_t \, dt + \frac{1}{2} f_{xx} dS_t \cdot dS_t + f_x dS_t$$

$$= \{f_t + \frac{1}{2} f_{xx} \sigma^2 S_t^2 + f_x (\mu - r_D) S_t\} \, dt + f_x \sigma S_t \, dB_t.$$

Matching coefficients, we first find $a_t = f_x(t, S_t)$ from the $dt$ terms and then from the $dB_t$ terms we find

$$b_t = \frac{1}{r \beta_t} \{f_t + \frac{1}{2} f_{xx} \sigma^2 S_t^2 - r_D a_t S_t\}.$$

When we substitute these values into $f(t, S_t) = V_t = a_t S_t + b_t \beta_t$ we get

$$f(t, S_t) = f_x(t, S_t) S_t + \frac{1}{r} (f_t(t, S_t) + \frac{1}{2} \sigma^2 f_{xx}(t, S_t) S_t^2 - r_D f_x(t, S_t) S_t).$$

For this to hold for all $S_t$ we need for all $x$ that

$$f_t(t, x) = -\frac{1}{2} \sigma^2 x^2 f_{xx}(t, x) - (r - r_D) x f_x(t, x) + r f(t, x).$$