Yield Curve Models

and Their Applications

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Outline

- Bond Models in General and HJM in Particular
- A Bond Ratio Factorization
- The Constant Sign Condition
- Inequalities for Futures Rates and Swap Rates
- Empirical Analyses of Futures and Swaps
- A More Apt Foundation Interest Rate Futures
- Concluding Remarks
Bonds: Requirements for any Model

• Default-free Zero-coupon Bond

By a zero-coupon bond of maturity $T$ we mean a security that is certain to pay to its holder one unit of cash at a date $T$.

• Basic Properties

If the price of a zero-coupon bond of maturity $T$ at time $t$ is denoted by $P(t,T)$, then $P(t,T)$ satisfies

1. $P(t,T) > 0$ for any $0 \leq t \leq T$,

2. $P(T,T) = 1$, and

3. $P(t,T) \leq 1$ for any $0 \leq t \leq T$. 
The Heath-Jarrow-Morton Bond Price Model

- **Bond Prices**
  \[ P(t, T) = \exp \left( - \int_t^T f(t, u) \, du \right) \quad 0 \leq t \leq T \leq \tau \quad (1) \]

- **Properties of the Bond Process** \( P(t, T) \)
  1. \( P(t, T) > 0 \),
  2. \( P(T, T) = 1 \),
  3. if \( f(\cdot, \cdot) \geq 0 \), then \( P(t, T) \leq 1 \),
  4. for any fixed \( T \) the bond process \( P(\cdot, T) \) is adapted to \( \{ \mathcal{F}_t \} \), and
  5. \( P(t, T) \) is differentiable with respect to \( T \).

- **Forward Rate Process**
  \[ f(t, u) = f(0, u) + \int_0^t \alpha(s, u) \, ds + \int_0^t \sigma(s, u) \, dB_s, \quad (2) \]

  where \( B_s \) denotes an \( n \)-dimensional Brownian motion and where
  \( \{ \alpha(s, u) : 0 \leq s \leq u \leq \tau \} \) and \( \{ \sigma(s, u) : 0 \leq s \leq u \leq \tau \} \)
  are respectively \( \mathbb{R} \) and \( \mathbb{R}^n \) valued measurable processes that are adapted to the standard filtration \( \mathcal{F}_s \) of \( \{ B_s \} \).

- **The Spot Rate and the Accumulation Factor**
  \[ r(t) = f(t, t) \quad \text{and} \quad \beta(t) = \exp \left( \int_0^t r(u) \, du \right) \quad (3) \]
A Gallery of Classical Models

Bond Price and Forward Rate Models:

\[ P(t, T) = \exp \left( -\int_t^T f(t, u) \, du \right) \quad 0 \leq t \leq T \leq \tau \]  \hspace{1cm} (4)

\[ f(t, u) = f(0, u) + \int_0^t \alpha(s, u) \, ds + \int_0^t \sigma(s, u) dW_s, \]  \hspace{1cm} (5)

Examples — One and Two Factor Models

- **Continuous Ho-Lee Model:** \( \sigma(\omega, t, T) = \sigma > 0 \)

- **Vasicek Model:** \( \sigma(\omega, t, T) = \sigma \exp(-\delta(T - t)), \quad \sigma, \delta > 0 \)

- **Square Root Model:** \( \sigma(\omega, t, T) = \sigma |f(t, T)|^{1/2}, \quad \sigma > 0 \)

- **Proportional Model:** \( \sigma(\omega, t, T) = \sigma \min(f(t, T), \delta), \quad \sigma, \delta > 0 \)

- **2-Factor Combination: Ho-Lee plus Vasicek Models**

  \[ \sigma_1(\omega, t, T) = \sigma_1, \quad \sigma_1 > 0 \]

  \[ \sigma_2(\omega, t, T) = \sigma_2 \exp(-\delta(T - t)), \quad \sigma_2, \delta > 0 \]
Absence of Arbitrage
and Existence of an Equivalent Martingale Measure

• What must we require of the instantaneous forward rate \( f(t, T) \) to guarantee the absence of arbitrage profits?

• By classic results of arbitrage pricing theory, one can show there are no arbitrage profits provided that there is an equivalent martingale measure, a probability measure \( \tilde{P} \) with the same null sets as \( P \) such that \( \{P(t, T)/\beta(t) : 0 \leq t \leq T\} \) is a martingale with respect to the filtration \( F_t \) for each fixed \( T \).

• Forward Rate Drift Restriction
Heath, Jarrow, and Morton (1992) established the existence of such a measure provided that the drift \( \alpha(t, T) \) and the volatility \( \sigma(t, T) \) satisfy the forward rate drift restriction

\[
\alpha(t, T) = \sigma(t, T) \perp \left[ \gamma(t) + \int_t^T \sigma(t, u) du \right]
\]  

(6)

where \( \gamma(t) \) is an adapted \( n \)-dimensional process such that

\[
E \exp \left( -\int_0^\tau \gamma(u) \perp dB_u - \frac{1}{2} \int_0^\tau |\gamma(u)|^2 \, du \right) = 1.
\]  

(7)
Consequence of the HJM Assumptions

- **Equivalent Martingale Measure**

  \[ \tilde{P}(A) = E \left[ 1_A \exp \left( - \int_0^T \gamma(u) dB_u - \frac{1}{2} \int_0^T |\gamma(u)|^2 du \right) \right] \]  
  \[ \text{(8)} \]

  For each \( T \) the discounted process \( \{ P(t, T) / \beta(t) : 0 \leq t \leq T \} \) is a \( \tilde{P} \)-martingale with respect to the filtration \( \mathcal{F}_t \).

- Under \( \tilde{P} \) the process

  \[ \tilde{B}_t = B_t + \int_0^t \gamma(u) \, du \]  
  \[ \text{(9)} \]

  is a standard Brownian motion.

- **Bond Dynamics under the Equivalent Martingale Measure**

  \[ dP(t, T) = P(t, T) \left[ r(t) \, dt + a(t, T) \, d\tilde{B}_t \right], \]  
  \[ \text{(10)} \]

  where \( a(t, T) \) is the \( n \)-dimensional column vector of integrated volatilities defined by

  \[ a(t, T) = - \int_t^T \sigma(t, u) \, du, \]  
  \[ \text{(11)} \]

  so \( P(t, T) \) has the integral representation:

  \[ \beta(t) P(0, T) \times \exp \left[ \int_0^t a(s, T) \, d\tilde{B}_s - \frac{1}{2} \int_0^t |a(s, T)|^2 \, ds \right], \]

  where \( P(0, T) \) is the initial yield curve.
The Driving Force: A Bond Ratio Factorization

For $0 \leq t \leq T \leq U$ one has the factorization

$$\frac{P(t, T)}{P(t, U)} = \frac{P(0, T)}{P(0, U)} \eta(t, T, U) \xi(t, T, U)$$

where

$$\eta(t, T, U) = \exp \left[ \int_0^t [a(s, T) - a(s, U)] \, d\tilde{B}_s - \frac{1}{2} \int_0^t |a(s, T) - a(s, U)|^2 \, ds \right],$$

and

$$\xi(t, T, U) = \exp \left[ - \int_0^t a(s, U)^\perp [a(s, T) - a(s, U)] \, ds \right].$$

The Constant Sign Condition

If $\{P(t, T) : 0 \leq t \leq T \leq \tau\}$ is an HJM family of bond prices for which all components of $\sigma(t, T)$ are nonnegative, then we say the family satisfies the constant sign condition.

A Very Useful Submartingale

When the constant sign condition is satisfied,

$$M_t = P(t, U)/P(t, T)$$

is a $P$ submartingale.
• **LIBOR Quotes**

By convention, LIBOR (London Inter-Bank Offered Rate) quotes are stated as *add-on yields*. For example, if \( L = 0.06 \) is the current LIBOR quote for a 90 day deposit, then the investor who commits \( M = 100 \) dollars now will receive after 90 days the return of the principle and an interest payment of

\[
M \cdot L \cdot (90/360) = 100 \cdot 0.06 \cdot (90/360) = 1.5 \text{ dollars}.
\]

We scale time so that for any two times \( s \) and \( t \) the number of calendar days between these times is \( 360(t - s) \).

We will also let \( L_\lambda(t) \) denote the LIBOR quote at time \( t \) for a deposit of \( 360\lambda \) days, so \( \lambda = 1/4 \) corresponds to a 90-day term of deposit.

• **Link between LIBOR Quotes and Zero-coupon Bonds**

\[
\lambda L_\lambda(t) = \frac{1}{P(t, t + \lambda)} - 1 
\]

Hence, we have

\[
L_\lambda(t) = \frac{1}{\lambda} \left( \frac{1}{P(t, t + \lambda)} - 1 \right), \quad \text{or} \quad P(t, t + \lambda) = \frac{1}{1 + \lambda L_\lambda(t)}.
\]
Chickens, Eggs, and Futures Contracts

• For the moment, we consider a GENERAL futures price $\phi_t$ at time $t$ to be determined *exogenously*. We will then find the relationships that drive $\phi_t$.

• Rules of a Futures Contract started at time $t \leq T$

  1. No cash changes hands at time $t$.

  2. Each subsequent day the accounts of the contracting parties are adjusted by the change in $\phi_t$.

  3. The accounts are modelled as being interest-bearing.

  4. At the terminal time $T$, the futures price $\phi_T$ is determined by a contractual agreement.

• Finally, the futures price $\phi_t$ for $t \in [0, T]$ is actually governed by the supply and demand for futures contracts. There is no cost to enter into such contracts, but the the futures price $\phi_t$ determines the attractiveness of the agreement.
Futures Contracts: Assumptions of the Theory

- **Futures Price.** The futures price process is assumed to be well-modelled by some semimartingale $\phi_t$.

- **Cash Flows.** The cash flow to (or from) a person who takes a long position between dates $s$ and $t$ nets out to $\phi_t - \phi_s$.

- **Price of the Futures Contract.** The arbitrage price of the cash flow determined by a futures contract is always equal to zero.

- **Terminal Condition.** $\phi_T = B$, where $B$ is an $\mathcal{F}_T$-measurable random variable.

Finally, under *modest regularity*, one can show that any $\{\phi_t : t \geq 0\}$ meeting these conditions will satisfy

$$\phi_t = \tilde{E}[B|\mathcal{F}_t].$$
LIBOR Futures Prices (and Rates)

We let $F^p_{\lambda}(t, T)$ denote the LIBOR futures price at time $t$ that is associated with a futures contract that is based on a LIBOR deposit of maturity $\lambda$ and with a settlement date $T$.

- **Terminal condition.** The value of the LIBOR futures price at the settlement time $T$ is determined to be $100(1 - L_\lambda(T))$; so, $F^p_{\lambda}(t, T)$ satisfies the terminal condition

  $$F^p_{\lambda}(T, T) = 100(1 - L_\lambda(T)). \quad (13)$$

- **LIBOR Futures Price Process.** Under modest regularity conditions, the LIBOR futures price process is a $\tilde{P}$-martingale, and we have the representation

  $$F^p_{\lambda}(t, T) = \tilde{E}(100(1 - L_\lambda(T))|\mathcal{F}_t) \quad (14)$$

  so, in terms of the bond prices, we have

  $$F^p_{\lambda}(t, T) = 100 \left( \frac{1}{\lambda} + 1 \right) - \frac{100}{\lambda} \tilde{E} \left[ \frac{1}{P(T, T + \lambda)} \middle| \mathcal{F}_t \right]. \quad (15)$$
Three Facts in Prospective

- An HJM family of bonds \( \{P(t, T) : 0 \leq t \leq T \leq \tau\} \) has a representation for \( P(t, T) \) given by
  \[
  \beta(t)P(0, T) \times \exp \left[ \int_0^t a(s, T) d\tilde{B}_s - \frac{1}{2} \int_0^t |a(s, T)|^2 ds \right]
  \]

- LIBOR quotes are tied to the bond model by
  \[
  L_\lambda(t) = \frac{1}{\lambda} \left( \frac{1}{P(t, t+\lambda)} - 1 \right)
  \]

- LIBOR futures prices have the martingale representation
  \[
  F^p_\lambda(t, T) = \tilde{E}(100(1 - L_\lambda(T))|\mathcal{F}_t)
  = 100 \left( \frac{1}{\lambda} + 1 \right) - \frac{100}{\lambda} \tilde{E} \left[ \frac{1}{P(T, T + \lambda)} |\mathcal{F}_t} \right]
  \]

- LIBOR futures rates have the martingale representation
  \[
  F_\lambda(t, T) = \tilde{E}(L_\lambda(T)|\mathcal{F}_t)
  = \frac{1}{\lambda} \tilde{E} \left[ \frac{1}{P(T, T + \lambda)} - 1 \right|\mathcal{F}_t}
  \]
MAIN RESULT:
Futures Rate-Forward Rate Inequality

If \( \{P(t,T) : 0 \leq t \leq T \leq \tau\} \) is an HJM family of bond prices for which all components of \( \sigma(t,T) \) are nonnegative, then for all \( 0 < t < T < \tau \), the forward rate \( L_\lambda(t,T) \) and the futures rate \( F_\lambda(t,T) \) satisfy the inequality

\[
L_\lambda(t,T) \leq F_\lambda(t,T).
\]

(16)

Source and Consequences of the FRFR inequality:

- This bound follows as a direct consequence of the \( \tilde{P} \)-submartingale property of the process \( M_t = P(t,U)/P(t,T) \).

- The bound has concrete implications for the pricing of Eurodollar futures and for the interest rate swaps that are priced off of the Eurodollar futures.

- The bound holds for every HJM model, and therefore informs us about a whole class of models.

- The bound can be tested empirically.
Futures Price Inequality

Let \( \{P(t, T) : 0 \leq t \leq T \leq \tau\} \) denote an HJM family of bond prices that satisfy the constant sign condition; that is,

\[
\text{all components of } \sigma(t, T) \text{ are nonnegative.} \tag{17}
\]

The LIBOR quote \( L_\lambda(t) \) is linked to the HJM model by the simple bookkeeping relation

\[
L_\lambda(t) = \frac{1}{\lambda} \left( \frac{1}{P(t, t + \lambda)} - 1 \right), \tag{18}
\]

and the LIBOR futures price process \( F_\lambda(t, T) \) satisfies the fundamental martingale identity

\[
F_\lambda^p(t, T) = \tilde{E}(100(1 - L_\lambda(T))|\mathcal{F}_t), \tag{19}
\]

so, in theory, it must also satisfies the inequality

\[
F_\lambda^p(t, T) \leq 100 \left( \frac{1}{\lambda} + 1 \right) - \left( \frac{100}{\lambda} \right) \frac{1 + (T - t + \lambda)L_{T-t+\lambda}(t)}{1 + (T - t)L_{T-t}(t)}
\]
Components of an Empirical Test

• LIBOR quotes can be found on the website of the British Bankers' Association

  www.bba.org.uk

• Eurodollar futures prices are also publicly available from the website

  www.barchart.com/cme/cmedta.htm

• Special Nature of LIBOR quotes

  – Survey-based — not directly based on trades

  – Average vs. a Distribution

  – ... but the distribution is tight.
Implementation of and Empirical Test

- As an illustration, let’s compute the value of the upper bound for January 2nd, 1998 for a 90-day LIBOR Eurodollar rate with settlement in September 1998.

- The upper bound for 90-day LIBOR futures prices is

\[ U_{1/4}(t, T) = 500 - 400 \frac{1 + (T - t + 1/4)L_{T-t+1/4}(t)}{1 + (T - t)L_{T-t}(t)} , \]

where \( t \) and \( T \) are all scaled so that one unit equals 360 days.

- The settlement date for each contract is the second London business day before the third Wednesday of the contract month. For the contract that expires in September 1998 the settlement date \( T \) is 14th of September 1998, so \( T - t = 255/360 \).

- Finally, we need the values of the LIBOR quotes with the maturities \( \lambda = 255/360 \) and \( \lambda = 345/360 \), but observed LIBOR quotes are publicly available only for maturities \( \lambda \) given by \( 7/360, 30/360, \ldots, 360/360 \). Thus, an interpolation is needed, and we used the cubic spline interpolation of S-Plus.
90-Day LIBOR Futures Prices and Theoretical Upper Bound for the Twenty Trading Days During January 1998 for the Contract with Settlement in September 1998
Violations of the Theoretical Futures Price Upper Bound

<table>
<thead>
<tr>
<th>Month</th>
<th>Cts</th>
<th>TDs</th>
<th>Obs</th>
<th>%V</th>
<th>Median $\Delta$</th>
<th>MAD $\Delta$</th>
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<tr>
<td>Jan</td>
<td>4</td>
<td>20</td>
<td>80</td>
<td>66.25</td>
<td>0.0150</td>
<td>0.0307</td>
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<tr>
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<td>4</td>
<td>19</td>
<td>76</td>
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<td>0.0193</td>
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<td>0.0427</td>
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<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
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<td>21</td>
<td>126</td>
<td>80.95</td>
<td>0.0442</td>
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</tr>
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</table>

- Defect is measured by $F_\lambda(t, T) - U_\lambda(t, T)$

- "Obs" denotes the number of observed LIBOR futures prices

- "%V" is the percentage of times that the market price of the futures was larger than the theoretical upper bound
Swap Rates in Practice

The arbitrage free swap rate for a (plain vanilla) swap is completely determined by the forward rates. Unfortunately, the forward rates are not directly observable. In practice, futures rates are used as surrogates for the forward rates. This practice is known to be only an approximation, but its theoretical consequences have not been thoroughly explored.

Swap Rate Inequality

For any HJM family

\[ \{P(t, T) : 0 \leq t \leq T \leq \tau \} \]

of bond prices with drift rates that satisfy the constant sign condition, the swap rate \( \kappa_0 \) is bounded above by

\[
1 + \lambda L_{\lambda}(T_0) - \prod_{k=1}^{N-1}(1 + \lambda F_{\lambda}(T_0, T_k))^{-1} \\
\frac{\lambda \left[1 + (1 + \lambda F_{\lambda}(T_0, T_1))^{-1} + \cdots + \prod_{k=1}^{N-1}(1 + \lambda F_{\lambda}(T_0, T_k))^{-1}\right]}{\lambda [1 + (1 + \lambda F_{\lambda}(T_0, T_1))^{-1} + \cdots + \prod_{k=1}^{N-1}(1 + \lambda F_{\lambda}(T_0, T_k))^{-1}]}.
\]
Box plots of the difference between theoretical upper bounds and the observed swap rates for all trading days of the period July 2000 - June 2001 and maturities of 1-7 years.

- For long maturities, the gap is mainly positive and the gap widens with maturity as theory suggests.

- For swaps with short maturities there are many days when the observed difference is negative — at variance with theory.
Box plots of the gap between the theoretical upper bounds and the observed 1-year swap rates month by month for the period July 2000 - June 2001.

- Violations are common and substantial.
- The month of November 2000 was particularly extreme.
- The predominant gap size is less than 5 basis points.
- For negative gaps the size averages about 3 basis points.
Four Elements Toward a Resolution of the Empirical Puzzles

• Perhaps, there is no HJM model that is adequate for the analysis of LIBOR futures.

• Perhaps, there are arbitrage opportunities in the market of LIBOR futures.

• Perhaps, there are data anomalies.

• Perhaps, transaction costs or other market frictions render the apparent arbitrage opportunity as simply a mirage.
Now: A Bonus Question Regarding the Martingale Representation of Futures Prices

• For many HJM models the spot rate may be unbounded. For example, the continuous Ho-Lee model has the spot rate

\[ r(t) = f(0, t) + \sigma^2 t^2 / 2 + \sigma \tilde{B}_t. \]

• Two questions.

1. How one can replace the boundedness condition on the spot rate?

2. How one can replace the assumption that \( F_\lambda(t, T) \) satisfies the martingale identity

\[ F_\lambda(t, T) = \tilde{E} \left[ 100(1 - L_\lambda(T)) \mid \mathcal{F}_t \right]. \]
Some Background Observations

• “Fair” Price of Payoff $X$
  \[
  \tilde{E} \left[ \frac{\beta(t)}{\beta(T)} X \bigg| \mathcal{F}_t \right]
  \]
  \text{present value}

• “Fair” Price of Cash Flows Associated with Futures Price $\phi(\cdot)$
  \[
  \tilde{E} \left[ \int_t^T \frac{\beta(t)}{\beta(s)} d\phi(s) \bigg| \mathcal{F}_t \right].
  \]
  \text{the present value of payoff } d\phi(s)

• “The price of the futures contract is always zero” means
  \[
  \tilde{E} \left[ \int_t^T \beta(s)^{-1} d\phi(s) \bigg| \mathcal{F}_t \right] = 0, \quad \text{for all } t.
  \]
  Hence, the process
  \[
  I(t) = \int_0^t \beta(s)^{-1} d\phi(s)
  \]
  is a $\tilde{P}$-martingale.

• If the spot rate $r(\cdot)$ is bounded on $\Omega \times [0, \tau]$, then the process
  $I(\cdot)$ is a $\tilde{P}$-martingale “if and only if” the process $\phi(\cdot)$ is a $\tilde{P}$-martingale.

• Since $\phi(T) = B$, we have
  \[
  \phi(t) = \tilde{E}[B|\mathcal{F}_t].
  \]
A More Appropriate
Interest Rate Futures Martingale Representation

• One replaces the assumption that the LIBOR futures price process is \( \tilde{P} \)-martingale by two more “technical” assumptions

  1. the futures price process with terminal price \( 100(1-L_\lambda(T)) \) exists, and

  2. the futures price process has an integrable with respect to \( \tilde{P} \) quadratic variation.

• To show that the process \( F^p_\lambda(t,T) \) defined by

\[
F^p_\lambda(t,T) = \tilde{E}(100(1-L_\lambda(T))|\mathcal{F}_t)
\]

is in fact a futures price process we need to check two basic conditions

\[
\tilde{E}\left[ 100(1-L_\lambda(T)) \right]^2 < \infty,
\]

and

\[
\tilde{E} \int_0^T \beta(s)^{-2} d\langle F_\lambda(\cdot, T) \rangle(s) < \infty.
\]
Bottom Line: What Has Be Found?

- Arbitrage pricing theory has been used to derive a simple submartingale property of the bond price ratios that holds uniformly for a large class of HJM models.

- This yields A Priori inequalities for LIBOR futures prices and for the arbitrage price of swaps.

- These bounds are often violated in both markets, and the violations may be large enough to be economically significant.

- As a bonus one obtains a pointer toward the “right hypotheses” for framing work with interest rate futures. The suggested hypothesis can accommodate a Gaussian spot rate and other unbounded spot rates which more usual assumption systems cannot.