GAMBLING TEAMS AND WAITING TIMES FOR PATTERNS IN TWO-STATE MARKOV CHAINS

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Abstract. Methods using gambling teams and martingales are developed and applied to find formulas for the expected value and the generating function of the waiting time until one observes one of the elements of a finite collection of patterns in a sequence which is generated by a two-state Markov chain. (Key-words: Gambling, teams, waiting times, patterns, success runs, failure runs, Markov chains, martingales, stopping times, generating functions. Mathematics Subject Classification (2000): Primary 60J10, Secondary 60G42)

1. Introduction

How long must one observe a stochastic process with values from a finite alphabet until one sees a realization of a pattern which belongs to a specified collection $C$ of possible patterns? For independent processes this is an old question; in some special cases it is even considered by Feller (1968). Nevertheless, in the context of more general processes, or even for Markov chains, there are many natural problems which
have not been fully addressed. The main goal here is to show how some progress can be made by further developing the martingale methods which were introduced by Li (1980) and Li and Gerber (1981) in their investigation of independent sequences. Their key observation was that information on the occurrence times of patterns can be obtained from the values assumed by a specially constructed auxiliary martingale at a certain well-chosen time.

In the case of (first-order or higher-order) Markov chains, this observation is still useful, but to make it work requires a rather more elaborate plan for the construction of the auxiliary martingale. This construction depends in turn on several general devices which seem more broadly useful; these include “teams of gamblers,” “watching then betting,” “reward matching,” and a couple of other devices which will be described shortly.

Before engaging that description, we should note that pattern matching has been studied by many other techniques. The combinatorial methods of Guibas and Odlyzko (1981a, 1981b) are particularly effective, and there are numerous treatments of pattern matching problem by probabilistic techniques, such as Blom and Thorburn (1982), Breen et al. (1985), Chrysaphinou and Papastavridis (1990), Han and Hirano (2003), Robin and Daudin (1999), and Uchida (1998). One of the more general techniques is the Markov chain embedding method introduced by Fu (1986) which has been further developed in Antzoulakos (2001), Fu (2001), Fu and Chang (2002), and Fu and Koutras (1994). Only a few investigations considered waiting time problems for higher order Markov chains, and these have all focused on specific waiting times such as the “sooner or later” problem of Aki et al. (1996).
2. Expected Waiting Time Until a Pattern is Observed

We take \( \{Z_i, i \geq 1\} \) to be a Markov chain with two states \( S \) and \( F \), which may model “success” and “failure.” We suppose the chain has the initial distribution \( P(Z_1 = S) = p_S, P(Z_1 = F) = p_F \) and the transition matrix

\[
\begin{pmatrix}
p_{SS} & p_{FS} \\
p_{SF} & p_{FF}
\end{pmatrix},
\]

where \( p_{SF} \) is shorthand for \( P(Z_{n+1} = F|Z_n = S) \). We then consider a collection \( C \) of finite sequences \( A_j, 1 \leq j \leq K \), from the two-letter alphabet \( \{S, F\} \). If \( \tau_{A_j} \) denotes the first time until the pattern \( A_j \) has been observed as a completed run in the series \( Z_1, Z_2, \ldots \), then the random variable of main interest here is \( \tau_C = \min\{\tau_{A_1}, \ldots, \tau_{A_K}\} \), the first time when we observe a pattern from \( C \). Throughout our discussion we assume that no pattern of \( C \) contains another pattern from \( C \) as an initial segment. Naturally, this assumption entails no loss of generality.

A Run of “Failures” Under a Markov Model

To illustrate the construction, we first consider the rather easy case where \( K = 1 \) and were the pattern \( A_1 \) is a run of \( r \) consecutive \( F \)s. Thus, we will compute the expected value of \( \tau = \tau_{A_1} \), the time of the first completion of a run of \( r \) “failures” under our two-state Markov model. This example can be handled by several methods, and it offers a useful benchmark for more challenging examples.

We consider a casino where gamblers may bet in successive rounds on the output of our given two-state Markov chain, and we assume that the casino is fair in sense which we will soon make precise. We then consider a sequence of gamblers, one of whom arrives just before each new round of betting. Thus, gambler number \( n + 1 \)
arrives in time to observe the result of the \( n \)th trial, \( Z_n \), and we assume that he bets a dollar on the event that next trial yields an \( F \). If \( Z_{n+1} = S \), he loses his dollar and leaves the game. If he is lucky and \( Z_{n+1} = F \), then he wins \( 1/p_{SF} \) when \( Z_n = S \) and he wins \( 1/p_{FF} \) when \( Z_n = F \). This is the sense in which the casino is fair; the expected return on a one dollar bet is one dollar.

After this gambler gets his money, he then bets his entire capital on the event that \( Z_{n+2} = F \). Again, if \( Z_{n+2} = S \), then the gambler leaves the game with nothing. On the other hand, If \( Z_{n+2} = F \), then the gambler wins this round, and his capital is increased by the factor \( 1/p_{FF} \). Successive rounds proceed in the same way, with a new gambler arriving at each new round and with the gamblers from earlier periods either continuing to win or else going broke and leaving.

Now we need to be precise about the end of this process. If gambler \( n + 1 \) begins by observing \( Z_n = S \), then he bets until either he goes broke or until he observes \( r \) successive \( F \)s, and, if gambler \( n + 1 \) begins by observing \( Z_n = F \), then bets until he either goes broke or until he observes \( r - 1 \) successive \( F \)s. Once some gambler stops without going broke, all of the gambling stops.

Finally, we let \( X_n \) denote the casino’s net gain at conclusion of round \( n \). Since each bet is fair and since the bet sizes depend only on the previous observations, the sequence \( X_n \) is a martingale with respect to the \( \sigma \)-field generated by \( \{Z_i, i \geq 1\} \).

Now we just need to consider the casino’s net gain \( X_\tau \) when the gambling stops. By calculating \( E(X_\tau) \) in two ways we will then obtain the expected value of the time \( \tau \).

At time \( \tau \) many gamblers are likely to have lost all their money; only those who entered the game after round number \( \tau - r - 1 \) have any money. We now face two
different ending scenarios. First, it could happen that we have a block (denoted by $F^{(n)}$) of $r$ instances of $F$ which occur a the very beginning of the sequence $\{Z_i, i \geq 1\}$. Second, we could happen that we end with $SF^{(r)}$, an $S$ followed by a block of $r$ instances of $F$. Obviously we do not need to consider the possibility of ending with $FF^{(r)} = F^{(r+1)}$ since by definition $F^{(r)}$ cannot occur before time $\tau$.

When we total up the wins and losses of all of the gamblers, we then find that the value of the stopped martingale $X_\tau$ is given exactly by

$$X_\tau = \begin{cases} 
\tau - 1 - \frac{1}{p_{FF}} - \frac{1}{p_{FF}^2} - \ldots - \frac{1}{p_{FF}^{r-1}}, & \text{1st scenario,} \\
\tau - 1 - \frac{1}{p_{SF}} - \frac{1}{p_{FF}^2} - \ldots - \frac{1}{p_{FF}^{r-1}}, & \text{2nd scenario,}
\end{cases}$$

which can be written more briefly as

$$X_\tau = \begin{cases} 
\tau - 1 - \frac{1 - p_{FF}^{r-1}}{p_{FF}^r (1 - p_{FF})}, & \text{1st scenario,} \\
\tau - 1 - \frac{1}{p_{SF} p_{FF}^{r-1}} - \frac{1 - p_{FF}^{r-1}}{p_{FF}^{r-1} (1 - p_{FF})}, & \text{2nd scenario.}
\end{cases}$$

Since $E[\tau] < \infty$ and since the increments of $X_n$ are bounded, so the optional stopping theorem for martingales tells us that $0 = E[X_1] = E[X_\tau]$. From this identity and the formula for $X_\tau$, algebraic simplification give us

$$(1) \quad E[\tau] = 1 + p_F \frac{1 - p_{FF}^{r-1}}{(1 - p_{FF})} + (1 - p_{FF} p_{FF}^{r-1}) \left( \frac{1}{p_{SF} p_{FF}^{r-1}} + \frac{1 - p_{FF}^{r-1}}{p_{FS} p_{FF}^{r-1}} \right).$$

**SECOND STEP: A SINGLE PATTERN**

We now consider the more subtle case of a single (non-run) pattern $A$ with length $r$, and for specificity we assume that the pattern begins with $F$, so $A = FB$ where we have $B \in \{S, F\}^{r-1}$. As before we consider a sequence of gamblers, but this time we need to consider three different ending scenarios:

(1) $A$ occurs at the beginning of the sequence $\{Z_i, i \geq 1\}$, or
(2) the pattern $SA$ occurs, or

(3) the pattern $FA$ occurs.

The probability $p_1$ of the first scenario is trivial to compute, but one then runs into trouble. We do not know probability that the pattern $SA$ will appear earlier than $FA$, so the probabilities of the second and third ending scenarios are not readily computed. To circumvent this problem we introduce two teams of gamblers.

**Rules for the Gambling Teams**

(1) A gambler from the first team who arrives before round $n$ watches the result of the $n^{th}$ trial, and then bets $y_1$ dollars on the first letter in the sequence $A$. If he wins he then bets all of his capital on the next letter in the sequence $A$, and he continues in this way until he either loses his capital or he observes all of the letters of $A$. Such players are called *straightforward gamblers*.

(2) The gamblers of the second team make use of the information that they observe. If gambler $n + 1$ observes $Z_n = S$ just before he begins his play, then he bets just like a straightforward gambler except that he begins by wagering $y_2$ dollars on the first letter of pattern $A$. On the other hand, if he observes $Z_n = F$ when he first arrives, then wagers $y_2$ dollars on the first letter of the pattern $B$. He then continues to wager on the successive letters of $B$ either until he loses or until he observes $B$. Such players are called *smart gamblers*.

The two gambling teams continue their betting, until one team stops. At that time, all gambling stops, and we consider the wins and losses. Only those gamblers who enter the game after the time $\tau - r - 1$ will have any money and the amount we they have will depend on the ending scenario. If we let $W_{ij}y_i$ denote amount
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of money that team $i \in \{1, 2\}$ wins in scenario $j \in \{1, 2, 3\}$, then the values $W_{ij}$ are easy to compute, and in terms of these value of stopped martingale $X_\tau$ which represents the casino’s net gain is given by

$$X_\tau = \begin{cases} (y_1 + y_2)(\tau - 1) - y_1 W_{11} - y_2 W_{21}, & \text{1st scenario,} \\ (y_1 + y_2)(\tau - 1) - y_1 W_{12} - y_2 W_{22}, & \text{2nd scenario,} \\ (y_1 + y_2)(\tau - 1) - y_1 W_{13} - y_2 W_{23}, & \text{3rd scenario.} \end{cases}$$

Now, if we take $(y_1^*, y_2^*)$ to be a solution of the system

$$y_1^* W_{12} + y_2^* W_{22} = 1, \quad y_1^* W_{13} + y_2^* W_{23} = 1,$$

we see that with these bet sizes we have a very simple formula for $X_\tau$:

$$X_\tau = \begin{cases} (y_1^* + y_2^*)(\tau - 1) - y_1^* W_{11} - y_2^* W_{21}, & \text{1st scenario,} \\ (y_1^* + y_2^*)(\tau - 1) - 1, & \text{2nd scenario,} \\ (y_1^* + y_2^*)(\tau - 1) - 1, & \text{3rd scenario.} \end{cases}$$

The optional stopping theorem then gives us

$$0 = (y_1^* + y_2^*)(E[\tau] - 1) - p_1(y_1^* W_{11} + y_2^* W_{21}) - (1 - p_1),$$

where $p_1$ is the probability of scenario one. We therefore find

$$E[\tau] = 1 + \frac{p_1(y_1^* W_{11} + y_2^* W_{21}) + (1 - p_1)}{y_1^* + y_2^*}.$$

The formula (2) is more explicit than it may seem at first. In the typical case, the calculation of $p_1$, $\{W_{ij} : 1 \leq i \leq 2, 1 \leq j \leq 3\}$ and $\{y_k^* : 1 \leq k \leq 2\}$ is genuinely routine, as one can see by the next example.

EXAMPLE: WAITING TIME UNTIL FSF

Here our straightforward gamblers bet $y_1$ dollars on $FSF$ without regard of the preceding observation. On the other hand, the smart gamblers bet $y_2$ dollars on
FSF if they observed $S$ before placing their first bet, but they bet $y_2$ dollars on $SF$ if they observed $F$. The three ending scenarios are now either $FSF$ at the beginning (scenario one), or one ends with $SFSF$ (scenario two), or one ends with $FFSF$ (scenario three). The $2 \times 3$ “profit matrix” $\{W_{ij}\}$ is then given by

$$
\begin{pmatrix}
\frac{1}{pSF} & \frac{1}{pFS} + \frac{1}{pSF} & \frac{1}{pFS} + \frac{1}{pSF} + \frac{1}{pSF} \\
\frac{1}{pFPFSF} + \frac{1}{pSF} & \frac{1}{pFSF} + \frac{1}{pFSF} + \frac{1}{pFSF} & \frac{1}{pFSF} + \frac{1}{pFSF} + \frac{1}{pFSF} \\
\frac{1}{pFFFSF} + \frac{1}{pSF} & \frac{1}{pFSF} + \frac{1}{pFSF} + \frac{1}{pFSF} & \frac{1}{pFSF} + \frac{1}{pFSF} + \frac{1}{pFSF}
\end{pmatrix},
$$

and bet sizes $y_1^*$ and $y_2^*$ are determined by the relations

$$
y_1^* \left( \frac{1}{pSF} + \frac{1}{pSF} \right) + y_2^* \left( \frac{1}{pFS} + \frac{1}{pSF} + \frac{1}{pSF} \right) = 1,
y_1^* \left( \frac{1}{pFPFSF} + \frac{1}{pSF} \right) + y_2^* \left( \frac{1}{pFSF} + \frac{1}{pFSF} \right) = 1,
$$

which one can solve to obtain

$$
y_1^* = \frac{pFFFSF}{pFS + pSF + pFSF}, \quad y_2^* = \frac{pFSF(pSF - pFF)}{pFS + pSF + pFSF}.
$$

The probability $p_1$ of the first scenario is just $pFPFSF$, so after substitution and simplification the general formula (2) provides

$$
E[\tau_{FSF}] = 1 + \frac{pS}{pSF} + \frac{1}{pSF} + \frac{1}{pFSF},
$$

which is as explicit as one could wish.

3. Expected Time Until One of Many Patterns

We now consider a collection $C = \{A_j : 1 \leq j \leq K\}$ of $K$ strings of possibly varying lengths from a finite alphabet, and we take on the task of computing the expected value of $\tau_C = \min\{\tau_{A_1}, \ldots, \tau_{A_K}\}$, the first time that one observes one of the patterns in $C$. The method we propose is analogous to the two-team method we just used, although many teams are now needed. The real challenge is the construction
of the list of the appropriate ending scenarios which now require some algorithmic considerations.

**Listing the Ending Scenarios**

Given $C = \{A_j\}_{1 \leq j \leq K}$ we first consider the set sequence transformation

$$C = \{A_j\}_{1 \leq j \leq K} \rightarrow \{SA_j, FA_j\}_{1 \leq j \leq K} = \{B_i\}_{1 \leq i \leq 2K} = C',$$

which doubles the cardinality of $C$. We then delete from $C'$ each pattern $B$ which can only occur after the stopping time $\tau_C$. The resulting collection $C''$ is called the **final list**. We denote the elements of $C''$ by $C_j$, $1 \leq j \leq K''$, and we note that $K \leq K' \leq 2K$.

To illustrate the construction, suppose the initial collection is $C = \{FSF, FF\}$. The **doubling step** gives us $C' = \{SFSF, FFSF, SFF, FFF\}$. Since $FFS$ and $FFF$ cannot occur before $\tau$, these are eliminated from $C'$ and final list is simply $C'' = \{SFSF, SFF\}$. Similarly, if the initial collection is $C = \{FS, SSS\}$, then the final list is $C'' = \{SFS, FFS\}$.

Now, before we describe the ending scenarios, we need one further notion. If patterns $C$ and $C'$ in the final list $C''$ satisfy $C = SA$ and $C' = FA$ for some pattern $A \in \mathcal{C}$, then we say that $C$ and $C'$ are **matched**. Also, if $C$ and $C'$ are matched and $C = SA$ and $C' = FA$, then we say that $C$ and $C'$ are **generated** by $A$. Finally, even though there are many ending scenarios, they are of just three basic kinds.

1. There are $K$ scenarios where one observes an element of $C$ as an initial segment of the Markov sequence $\{Z_i, i \geq 1\}$.
(2) There is a scenario for each unmatched pattern from $C'' \setminus C$. We denote the number of these by $L$.

(3) There is a scenario for each matched pattern from $C'' \setminus C$. We denote the number of these by $2M$.

**From the Listing to the Teams**

For each scenario associated with unmatched pattern $C_j$ we introduce one team of straightforward gamblers who bet $y_j$ dollars on the pattern $A_i$ which generated $C_j$. For each pair of scenarios associated with matched patterns $C_p$ and $C_m$ which were generated by pattern $A_k$, we introduce two teams. One team bets $y_p$ dollars on $A_k$ in the straightforward way, another bets $y_m$ dollars on $A_k$ in the smart way. If $W_{ij}y_i, i = 1, 2$ denotes amount of money that the $i^{th}$ team wins in the $j^{th}$ scenario, then the stopped martingale $X_\tau$ is given by

$$X_\tau = \sum_{i=1}^{L+2M} y_i (\tau - 1) - S(y_1, ..., y_{L+2M}),$$

where we have set

$$S(y_1, ..., y_{L+2M}) = \sum_j 1_{E_j} \sum_{i=1}^{L+2M} y_i W_{ij},$$

where $1_{E_j}$ is the indicator function for the even $E_j$ that the $j^{th}$ scenario occurs.

If $(y_1^*, ..., y_{L+2M}^*)$ is a solution of the linear system

$$y_1^* W_{1 \, K+1} + \cdots + y_{L+2M}^* W_{L+2M \, K+1} = 1,$$

$$\vdots$$

$$y_1^* W_{1 \, K+L+2M} + \cdots + y_{L+2M}^* W_{L+2M \, K+L+2M} = 1,$$

(3)
then we have
\[
S(y_1^*, ..., y_{L+2M}^*) = \begin{cases} 
\sum_{i=1}^{L+2M} y_i^* W_{ij}, & \text{in scenario } j \in \{1, 2, ..., K\} \\
1, & \text{in scenario } j > K 
\end{cases}
\]

By the optional stopping theorem we have
\[
0 = \mathbb{E}[X_1] = \mathbb{E}[X_{\tau_C}] = \sum_{i=1}^{L+2M} y_j^*(\mathbb{E}[\tau_C] - 1) - \sum_{j=1}^{K} p_j \sum_{i=1}^{L+2M} y_i^* W_{ij} - (1 - \sum_{j=1}^{K} p_j),
\]
where \( p_i \) is the probability that \( A_i \) is an initial segment of \( \{Z_i, i \geq 1\} \). We can now solve this equation to obtain a formula for \( \mathbb{E}[\tau_C] \) which we summarize as a theorem.

**Theorem 1.** If \((y_1^*, y_2^*, ..., y_{L+2M}^*)\) solves the linear system (3), then
\[
\mathbb{E}[\tau_C] = 1 + \frac{\sum_{j=1}^{K} p_j \sum_{i=1}^{L+2M} y_i^* W_{ij} + (1 - \sum_{j=1}^{K} p_j)}{\sum_{i=1}^{L+2M} y_j^*}.
\]

As before, this formula is more explicit than it may seem at first. In particular example, all of the required terms can be computed straightforwardly in problems of modest size.

**Computation of the Profit Matrix**

Formula (4) requires one to compute the profit matrix \( \{W_{ij}\} \), and we will first show how this can be done in general. The method will then be applied a specific example to confirm that the formula (4) may be rewritten in terms of the basic model parameters.

Consider a scenario ends with pattern \( C = c_1c_2...c_m \in C'' \). The team of straight-forward gamblers who begin by betting one dollar and who bet on the successive terms of the pattern \( A = a_1a_2...a_p \) will by time \( \tau_C \) have won
\[
\min_{i=1}^{(m-1)p} \sum \delta_i^M(A, C),
\]
where \( \delta_{st}^i(A, C) = \frac{1}{p_{c_{m-i}, a_1} p_{a_1 a_2} \ldots p_{a_{i-1} a_i}} \) if \( a_1 = c_{m-i+1}, a_2 = c_{m-i+2}, \ldots, a_i = c_m \) and where \( \delta_{st}^i(A, C) = 0 \) otherwise. Similarly, the team of smart gamblers will have won

\[
\min(m-1, p) \sum_{i=1}^{\min(m-1, p)} \delta_{st}^{sm1}(A, C) + \sum_{i=1}^{\min(m-1, p-1)} \delta_{st}^{sm2}(A, C),
\]

where (1) we set \( \delta_{st}^{sm1}(A, C) = \frac{1}{p_{c_{m-i}, a_1} p_{a_1 a_2} \ldots p_{a_{i-1} a_i}} \) if

\[
a_1 = c_{m-i+1}, a_2 = c_{m-i+2}, \ldots, a_i = c_m \quad \text{and} \quad c_{m-i} \neq a_i
\]

and where we set \( \delta_{st}^{sm1}(A, C) = 0 \) otherwise and (2) we set \( \delta_{st}^{sm2}(A, C) \) to be equal to \( 1/p_{a_1 a_2} p_{a_2 a_3} \ldots p_{a_{i+1}} a_{i+1} \) when \( a_1 = c_{m-i}, a_2 = c_{m-i+1}, \ldots, a_{i+1} = c_m \) and set \( \delta_{st}^{sm2}(A, C) \) equal to zero otherwise.

**Explicit Determination of** \( E[\tau_C] \)

To illustrate the use of formula (4), we consider \( C = \{SS, FS\} \). After doubling and elimination find the final list \( \{FSS, SF, SF\} \), and we then need to work out the set of scenarios. We have two scenarios where \( C_1 = SS \) or \( C_2 = FS \) occur as an initial segment of \( \{Z_i : i = 1, 2, \ldots\} \). We also have the unmatched scenario \( C_3 = FSS \) associated with the pattern \( SS \), and we have a pair of matched scenarios \( C_4 = SF \) or \( C_5 FFSF \), which are associated with the pattern \( FSF \). The profit matrix \( \{W_{ij}\} \) is then given by

\[
\begin{pmatrix}
\frac{1}{PSS} & 0 & 0 \\
0 & \frac{1}{PSF} & \frac{1}{PSSF} + \frac{1}{PSS} \\
\frac{1}{PSSF} + \frac{1}{PSS} & 0 & 0 \\
0 & \frac{1}{PSSF} + \frac{1}{PSF} & \frac{1}{PSSF} + \frac{1}{PSF} + \frac{1}{PSF} \\
0 & \frac{1}{PSSF} + \frac{1}{PSF} & \frac{1}{PSSF} + \frac{1}{PSF}
\end{pmatrix}
\]
and, after solving the corresponding linear system, we find that the appropriate initial team bets are given by

\[ y_1^* = \frac{p_{FSS}}{1 + p_{FS}}, \quad y_2^* = \frac{p_{FF}p_{FSPSF}}{p_{FS} + p_{SF} + p_{FSPSF}}, \quad y_3^* = \frac{p_{FSPSF}(p_{SF} - p_{FF})}{p_{FS} + p_{SF} + p_{FSPSF}}. \]

The probabilities \( p_1 \) and \( p_2 \) that \( SS \) and \( FSF \) are initial segments of the process \( \{Z_i : i = 1, 2, \ldots\} \) are given by \( p_{SPSS} \) and \( p_{FPPFSF} \) respectively, so the formula (4) leads one to the pleasantly succinct result

\[ E[\tau_C] = 2 + p_{SPSF} + \frac{1 - p_{SPSS}}{p_{FS}}. \]

4. Generating Functions for Pattern Waiting Times

To find the generating function of the waiting time \( \tau \) we need to introduce the same scenarios and the same betting teams, but we need to make some changes in the design of the initial bets. A gambler from the \( i \textsuperscript{th} \) team who arrives at moment \( k - 1 \) and who begins his betting on round \( k \) will now begin with a bet of size \( y_i\alpha^k \) where \( 0 < \alpha < 1 \). If \( \alpha^\tau W_{ij}(\alpha)y_i \) denotes total winnings of the \( i \textsuperscript{th} \) team when the game ends with \( j \textsuperscript{th} \) scenario, then we call \( W_{ij}(\alpha) \) the \( \alpha \)-profit matrix. As before, the \( \alpha \)-profit matrix does not depend on \( \tau \), and it can be computed if we know the ending scenario.

If \( X_n \) again denotes the casino’s net gain at moment \( n \), then

\[ X_\tau = \frac{\alpha^2 - \alpha \alpha^\tau}{1 - \alpha} \sum_{i=1}^{K+2M} y_i - S(\alpha, y_1, \ldots, y_{L+2M}), \]

and we set

\[ S(\alpha, y_1, \ldots, y_{L+2M}) = \sum_j 1_{E_j} \sum_{i=1}^{K+2M} \alpha^\tau y_i W_{ij}(\alpha), \]

where, as before, \( 1_{E_j} \) is the indicator function for the even \( E_j \) that the \( j \textsuperscript{th} \) scenario occurs.
If \((y_1^*, \ldots, y_{L+2M}^*)\) is a solution of the linear system

\[
y_1^* W_1 K+1(\alpha) + \cdots + y_{L+2M}^* W_{L+2M} K+1(\alpha) = 1,
\]

then we might hope to mimic our earlier calculation of \(E[X_\tau]\), but unfortunately we run into trouble since \(E(\alpha^T I_{\{1^{th} \text{ scenario}\}})\) may not equal \(p_1 E\alpha^T\).

Nevertheless, if the \(j^{th}\) scenario occurs, then we know exactly the value of \(\tau\). It is equal to \(|A_j|\) — the length of \(j^{th}\) sequence. Therefore, we have a formula for the stopped martingale,

\[
X_\tau = \frac{\alpha^2 - \alpha \alpha^T}{1 - \alpha} \sum_{i=1}^{L+2M} y_i^* - \alpha^T - I(\alpha, y_1^*, \ldots, y_{L+2M}^*),
\]

where \(I(\alpha, y_1^*, \ldots, y_{L+2M}^*)\) is defined by setting

\[
I(\alpha, y_1^*, \ldots, y_{L+2M}^*) = \begin{cases} 
\alpha |A_j| \left[ \sum_{i=1}^{L+2M} y_i^* W_{ij}(\alpha) - 1 \right], & \text{in scenario } j \in \{1, 2, \ldots, K\} \\
0, & \text{in scenario } j > K.
\end{cases}
\]

From this formula, the optional stopping theorem, we then find an the anticipated formula for the moment generating function of \(\tau\).

**Theorem 2.** If \((y_1^*, \ldots, y_{L+2M}^*)\) is a solution of linear system (5), then one has

\[
E[\alpha^T] = \frac{\alpha^2}{1 - \alpha} \sum_{j=1}^{L+2M} y_i^* - \sum_{j=1}^{K} p_j \alpha |A_j| \left[ \sum_{i=1}^{L+2M} y_i^* W_{ij} - 1 \right] + \frac{\alpha}{1 - \alpha} \sum_{j=1}^{L+2M} y_i^*.
\]

**Computation of the \(\alpha\) Profit Matrix and a Concrete Example**

As before, one needs to know how to compute the profit matrix, before the formula (6) may be properly regarded as an explicit formula. This is only a little more difficult than before. First, assume that a scenario ends with the pattern...
\[ C = c_1c_2...c_m. \] The team of straightforward gamblers who bet a dollar on pattern \[ A = a_1a_2...a_p \] by the time \( \tau \) will win

\[
\min(m-1,p) \sum_{i=1}^{\min(m-1,p)} \delta^s_i(A, C) / \alpha^{i-1},
\]

while the team of smart gamblers will win

\[
\min(m-1,p) \sum_{i=1}^{\min(m-1,p-1)} \delta^s_{m1}(A, C) / \alpha^{i-1} + \min(m-1,p) \sum_{i=1}^{\min(m-1,p-1)} \delta^s_{m2}(A, C) / \alpha^{i-1}.
\]

These formulas provide almost everything we need, but to be completely explicit one needs to pass to an example.

**A Generating Function Example**

Consider the waiting time until one observes the 3-letter pattern \( FSF \) in the random sequence \( \{Z_n : n = 1, 2, ... \} \) produced by the Markov model. In this case, the \( \alpha \)-profit matrix \( \{W_{ij}\} \) is given by

\[
\begin{pmatrix}
\frac{1}{p_{SF}} & \frac{\alpha^{-1}}{p_{FS}p_{SF}} + \frac{1}{p_{SF}} \\
\frac{\alpha^{-2}}{p_{FF}p_{FS}p_{SF}} + \frac{1}{p_{SF}} & \frac{\alpha^{-2}}{p_{FS}p_{SF}} + \frac{\alpha^{-1}}{p_{FS}p_{SF}} + \frac{1}{p_{SF}} \\
\frac{\alpha^{-2}}{p_{FF}p_{FS}p_{SF}} + \frac{1}{p_{SF}} & \frac{\alpha^{-1}}{p_{FS}p_{SF}} + \frac{1}{p_{SF}}
\end{pmatrix},
\]

and by solving the associated linear system one finds

\[
y_1^* = \frac{\alpha^2 p_{FF} p_{FS} p_{SF}}{1 - \alpha p_{FF} + \alpha p_{SF} + \alpha^2 p_{FS} p_{SF}}, \quad y_2^* = \frac{\alpha^3 p_{FS} p_{SF} (p_{SF} - p_{FF})}{1 - \alpha (p_{SS} + p_{FF} - \alpha (p_{FF} - p_{SF}(1 - \alpha p_{SS})))}.
\]

The general moment generating representation (6) then gives us the simple formula

\[
E[\alpha^\tau] = \frac{\alpha^3 p_{FS} p_{SF} (p_{SF} + \alpha (p_{FS} - p_{SS}))}{1 - \alpha (p_{SS} + p_{FF} - \alpha (p_{FF} - p_{SF}(1 - p_{FS}(1 - \alpha p_{SS}))))}.
\]

Naturally, such a formula provides one with complete information on the distribution of \( \tau \). In this case, the simple rational form may be rewritten via partial fraction expansion to give an explicit formulas for \( P(\tau = k) \) via symbolic calculations.
5. Higher Order Markov Chains

The gambling team method has been discussed here only for two-state chains, and, for reasons which we explain later, this limitation is not easily lifted. Thus, it is particularly instructive that the gambling team method still deal effectively with second order chains two-state chains, though obviously one must avoid using the naive first order four-state representation for the second order chain.

In the team approach for a second order model the gamblers naturally observe two rounds of betting before they place their first bets. As a consequence, we must consider a larger number of final scenarios, and for each pattern $A = a_1 a_2 ... a_p$ we need to consider of seven cases. Three cases when pattern $A$ occurs at the initial cases, (1) $A$, (2) $SA$, or (3) $FA$, and four later cases where one observes (4) $SSA$ (5) $SFA$, (6) $FSA$, and (7) $FFA$.

Instead of a doubling step, one now has a quadrupling step where given the collection of patterns $C = \{A_j\}_{1 \leq j \leq K}$ perform the set sequence transformation

$$C = \{A_j\}_{1 \leq j \leq K} \rightarrow \{SSA_j, SFA_j, FSA_j, FFA_j\}_{1 \leq j \leq K} = \{B_i\}_{1 \leq i \leq 4K} = C',$$

which quadruples the cardinality of $C$. Elimination after the quadrupling step works just as before. Specifically, if a scenario can happen only after the stopping time $\tau$ it is deleted. The list $C''$ that remains is again called the final list.

Each pattern from the collection $C$ yields four or fewer ending scenarios, and for each pattern from $C$ produce four teams of gamblers each of which bet in its own way.
(1) A new gambler from the first team arrives two rounds before he can begin to
gamble, watches the results of two rounds, and then bets on the successive
letters pattern A, no matter what he saw on the first two rounds.

(2) A new gambler from the second team arrives two rounds before he can begin
to gamble, watches the results of two rounds. If he observes on these rounds
$S_1$ he bets on the sequence on $a_2a_3...a_p$, but if he observes anything other
than $S_1$ he places his bets according to A.

(3) A new gambler from the third team arrives two rounds before he can begin
to gamble, watches the results of two rounds. If he observes on these rounds
$F_1$ he bets on the sequence on $a_2a_3...a_p$, but if he observes anything
other than $F_1$ he places his bets according to A.

(4) Finally, a gambler from the fourth team bets on $a_3a_4...a_p$ if he observes
$a_1a_2$ and otherwise he bets on A.

Of course, if in the final list we have fewer than four ending scenarios associated
with pattern A we will introduce smaller number of teams.

The remainder of the analysis is analogous to the case of a first order chain. Since
we have matched the number of (non-initial) ending scenarios and the number of
teams, we see that by choosing the size of initial bet for each team appropriately
we can make all the expressions for the stopped martingale equal to 1, no matter
how the game ends.

6. Concluding Remark

A peculiar feature of the method developed here is that it only addresses
the waiting time problems of two-state chains. One can easily invent analogous
betting schemes for $n$-state chains, but so far these have not been effective. The
problem with the natural candidates is that they lead to situations where the number of ending scenarios is higher than a number of teams. As a result, one has too few free parameters to achieve the require matching.

A second way to approach the waiting time problems for an $n$-state chain is to try to use the theory of the two-state chain along with an encoding of $\{1, 2, \ldots, n\}$ into $n$ blocks of zeros and ones. This can be effective in some very special cases, but typically such an encoding does not lead one to a waiting time problem for a homogeneous two-state Markov chain. Thus, for $n$-state chains, the gambling team method seems somewhat problematical, even though for two-state chains it seems on many accounts to be the method of choice.

References


